Univerzitet u Zenici
Filozofski fakultet
Odsjek: Matematika i informatika
Akademska 2013/2014.

Sadržaj sveske sa vježbi iz
<u>Analize III</u>
(I dio sveske - sadrži gradivo od 1 do 7 sedmice)

Literatura i zbirke za dodatno usavršavanje:

- Vajzović, Malenica: Diferencijalni račun funkcija više promjenljivih;
- Vajzović, Malenica: Integralni račun funkcija više promjenljivih;
- Perić, Tomić, Karačić: Zbirka riješenih zadataka iz matematike II;
- Demidovič: Zadaci i riješeni primjeri iz više matematike s primjenom na tehničke nauke;
- Ljaško i ostali: Zbirka zadataka iz matematičke analize II;
- Miličić, Uščumlić: Zbirka zadataka iz više matematike II;
- Berman: Sbornik zadach po kursu matematicheskogo analiza;
- Fatkić, Dragičević: Diferencijalni račun funkcija dviju i više promjenjivih;

Sveska je skinuta sa stranice **pf.unze.ba/nabokov/**
Za uočene greške pisati na *infoarrt@gmail.com*

Dio tablice izvoda

1) $(c)'=0;$ 2) $(u+v-w)'=u'+v'-w';$

3) $(uv)'=u'v+v'u;$ 3a) $(cu)'=cu';$

4) $\left(\frac{u}{v}\right)'=\frac{u'v-v'u}{v^2};$ 4a) $\left(\frac{u}{c}\right)'=\frac{u'}{c};$

4б) $\left(\frac{c}{v}\right)'=-\frac{cv'}{v^2};$ 5) $(x^n)'=nx^{n-1};$

6) $(\sin x)'=\cos x;$ 7) $(\cos x)'=-\sin x;$

8) $(\operatorname{tg} x)'=\sec^2 x;$ 9) $(\operatorname{ctg} x)'=-\operatorname{cosec}^2 x.$

5) $(u^n)'=nu^{n-1}\cdot u';$ 8) $(\operatorname{tg} u)'=\sec^2 u\cdot u';$

6) $(\sin u)'=\cos u\cdot u';$ 9) $(\operatorname{ctg} u)'=-\operatorname{cosec}^2 u\cdot u'.$

7) $(\cos u)'=-\sin u\cdot u';$

10) $(a^u)'=a^u\ln a\cdot u';$ 11) $(\log u)'=\frac{u'}{u}\log e;$

10a) $(e^u)'=e^u u';$ 11a) $(\ln u)'=\frac{u'}{u};$

10б) $(a^x)'=a^x\ln a;$ 11б) $(\log x)'=\frac{1}{x}\log e;$

10в) $(e^x)'=e^x;$ 11в) $(\ln x)'=\frac{1}{x}.$

12) $(\arcsin u)'=\frac{u'}{\sqrt{1-u^2}};$ 12a) $(\arcsin x)'=\frac{1}{\sqrt{1-x^2}};$

13) $(\arccos u)'=-\frac{u'}{\sqrt{1-u^2}};$ 13a) $(\arccos x)'=-\frac{1}{\sqrt{1-x^2}};$

14) $(\operatorname{arc\,tg} u)'=\frac{u'}{1+u^2};$ 14a) $(\operatorname{arc\,tg} x)'=\frac{1}{1+x^2};$

15) $(\operatorname{arc\,ctg} u)'=-\frac{u'}{1+u^2};$ 15a) $(\operatorname{arc\,ctg} x)'=-\frac{1}{1+x^2}.$

Dio tablice integrala

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq-1.$

2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$
$=\ln|u|+C.$

3. $\int a^u\,du=\frac{a^u}{\ln a}+C;\ \int e^u\,du=$
$=e^u+C.$

4. $\int\sin u\,du=-\cos u+C.$

5. $\int\cos u\,du=\sin u+C.$

6. $\int\sec^2 u\,du=\operatorname{tg} u+C.$

7. $\int\operatorname{cosec}^2 u\,du=-\operatorname{ctg} u+C.$

8. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\operatorname{arc\,tg}\frac{u}{a}+C.$

9. $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int\frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+C.$

11. $\int\frac{du}{\sqrt{u^2+a}}=$
$=\ln\left|u+\sqrt{u^2+a}\right|+C.$

Razvijanje f-je u Furijeov red u intervalu $[a,b]$, $a<b$

Neka je $y=f(x)$ integrabilna f-ja na intervalu $[a,b]$. Brojeve

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$$

i $$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx, \quad n\in\mathbb{N}$$

nazivamo FURIJEOVI KOEFICIJENTI f-je $f(x)$.

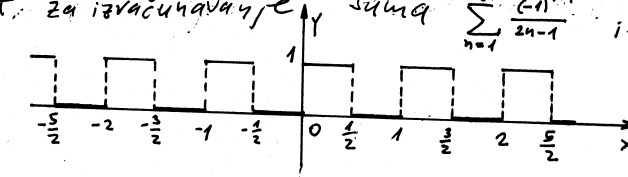
Trigonometrijski red

$$\frac{a_0}{2}+\sum\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right),\quad x\in[a,b]$$

se naziva FURIJEOV RED f-je $f(x)$.

Parnost i neparnost možemo ispitati samo u slučaju ako je interval $[a,b]$ simetričan u odnosu na nulu.

(#) Pretvoriti u Furijeov red f-ju definisanu grafikom. Iskoristiti dobijeni rezultat za izračunavanje suma $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$ i $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{2n-1}$.



R: F-ju predstavljenu grafikom označićemo sa $y=f(x)$.

F-ja je periodična perioda 1, što znači f-ju je dovoljno pretvoriti u Furijeov red u intervalu $[0,1]$.

Furijeovi koeficijenti na intervalu $[a,b]$ se računaju po formuli:

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\quad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx \quad i \quad b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$$

Interval $[0,1]$ nije simetričan u odnosu na 0, pa parnost i neparnost ne igraju nikakvu ulogu.

Furijeov red f-je $f(x)$ je oblika $\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$, $x\in[a,b]$

U našem slučaju:

$$a_0 = 2\int_0^1 f(x)\,dx = 2\int_0^{1/2} dx = 2\cdot\frac{1}{2}=1$$

$$a_n = 2\int_0^1 f(x)\cos 2n\pi x\,dx = 2\int_0^{1/2}\cos 2n\pi x\,dx = 2\,\frac{1}{2n\pi}\sin 2n\pi x\Big|_0^{\frac12} = \frac{1}{n\pi}\sin n\pi = 0$$

$$b_n = 2\int_0^1 f(x)\sin 2n\pi x\,dx = 2\int_0^{1/2}\sin 2n\pi x\,dx = 2\,\frac{1}{2n\pi}\left(-\cos 2n\pi x\Big|_0^{\frac12}\right) = \frac{(-1)}{n\pi}(\cos n\pi - 1)$$

$$= \frac{(-1)}{n\pi}\left((-1)^n-1\right) = \frac{1+(-1)^{n+1}}{n\pi}$$

$\boxed{2(2n-1)\pi x}$

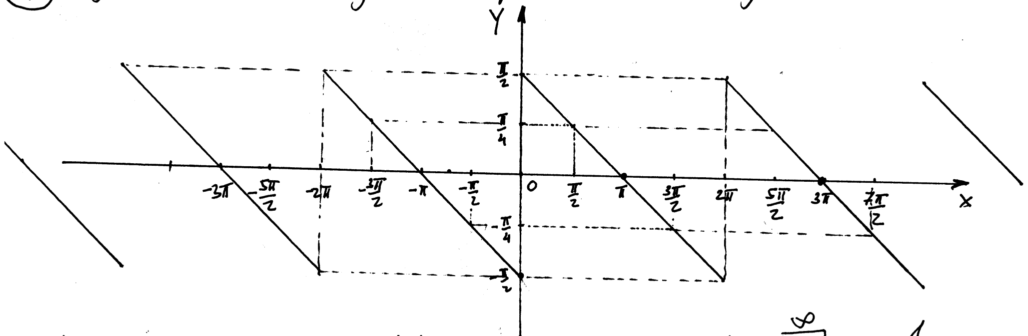
$$f(x) = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n\pi}\sin 2n\pi x = \frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{(2n-1)\pi}\sin 2(2n-1)\pi x$$

$$f(x) = \frac{1}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2(2n-1)\pi x}{(2n-1)}$$ f-ja razložena u Fourierov red

$f(\frac{1}{4})=1$ (iz grafika), $\sin 2(2n-1)\pi\cdot\frac{1}{4} = \sin(2n-1)\frac{\pi}{2} = (-1)^{n+1}$ pa je $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} = -\frac{\pi}{4}$ (ovaj rezultat se može dobiti na dva načina: U Furijeov red uvrstiti tačku $x=\frac{3}{4}$ ili prethodnu sumu pomnožiti sa (-1)).

\# F-ju definisanu grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n-3)}$.

Rj. Prvo primjetimo da je f-ja periodična što znači da se može pretvoriti u Furijeov red. Dalje, primjetimo da je period 2π što znači da možemo posmatrati npr. interval $[0, 2\pi]$.
F-ja na intervalu $[0,2\pi]$ prolazi kroz sljedeće tačke $(0,\frac{\pi}{2})$, $(\frac{\pi}{2},\frac{\pi}{4})$, $(\pi,0)$, $(\frac{3\pi}{2},-\frac{\pi}{4})$, $(2\pi,-\frac{\pi}{2})$. Jednačina prave kroz dvije tačke je

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1} \xrightarrow{\text{ako posmatramo}} \overset{x_1\ y_1}{(0,\tfrac{\pi}{2})} \text{ i } \overset{x_2\ y_2}{(\pi,0)} \qquad \frac{x-0}{\pi}=\frac{y-\frac{\pi}{2}}{-\frac{\pi}{2}}$$

$$\Rightarrow\ y-\frac{\pi}{2}=\frac{x}{\pi}\cdot\left(-\frac{\pi}{2}\right)\ \Rightarrow\ y-\frac{\pi}{2}=-\frac{x}{2}\ \Rightarrow\ y=\frac{\pi-x}{2}.$$

Furijeov red na intervalu $[a,b]$ je oblika

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

gdje se Furijeovi koeficijenti računaju po formuli $\qquad \frac{2n\pi x}{2\pi}=nx$

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\qquad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi}{b-a}\,dx$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\,dx=\frac{1}{2\pi}\left(\pi x\Big|_0^{2\pi}-\frac{1}{2}x^2\Big|_0^{2\pi}\right)=\frac{1}{2\pi}(2\pi^2-2\pi^2)=0$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\cos nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\cos nx\,dx=\left|\begin{matrix}u=\pi-x & dv=\cos nx\,dx\\ du=-dx & v=\frac{1}{n}\sin nx\end{matrix}\right|=$$

$$=\frac{1}{2\pi}\left(\frac{1}{n}(\pi-x)\sin nx\Big|_0^{2\pi}+\frac{1}{n}\int_0^{2\pi}\sin nx\,dx\right)=-\underbrace{\frac{1}{2n\pi}\sin 2n\pi}_{=0}+\frac{1}{2n\pi}\cdot\frac{-1}{n}\cos nx\Big|_0^{2\pi}=$$

$$=-\frac{1}{2n^2\pi}(\cos 2n\pi-1)=\frac{1}{2n^2\pi}(1-\cos 2n\pi)=\frac{1}{2n^2\pi}(1-1)=0$$

Sa grafika date f-je možemo primjetiti da je f-ja simetrična u odnosu na koordinatni početak tj. da je neparna, pa je $a_0=0$ i $a_n=0$ $\forall n\in\mathbb{N}$.

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\sin nx\,dx=\frac{1}{2\pi}\int_0^{2\pi}(\pi-x)\sin nx\,dx=\left|\begin{matrix}u=\pi-x & dv=\sin nx\,dx\\ du=-dx & v=-\frac{1}{n}\cos nx\end{matrix}\right|$$

$$=\frac{1}{2\pi}\left(-\frac{1}{n}(\pi-x)\cos nx\Big|_0^{2\pi}-\frac{1}{n}\int_0^{2\pi}\cos nx\,dx\right)=-\frac{1}{2n\pi}(-\pi\cos 2n\pi-\pi)-$$

$$-\frac{1}{2n\pi}\sin nx\Big|_0^{2\pi}=\frac{1}{2n}(\cos 2n\pi+1)=\frac{1}{2n}\cdot 2=\frac{1}{n}$$

Prema tome

$$f(x)\sim\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

Ako za x uzmemo $\frac{\pi}{2}$ imamo

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{\pi}{2}n=\sum_{k=1}^{\infty}\Bigg(\frac{1}{4k-1}\sin(4k-1)\frac{\pi}{2}+\frac{1}{4k-2}\underbrace{\sin(4k-2)\frac{\pi}{2}}_{\sin(2k-1)\pi=0}+$$

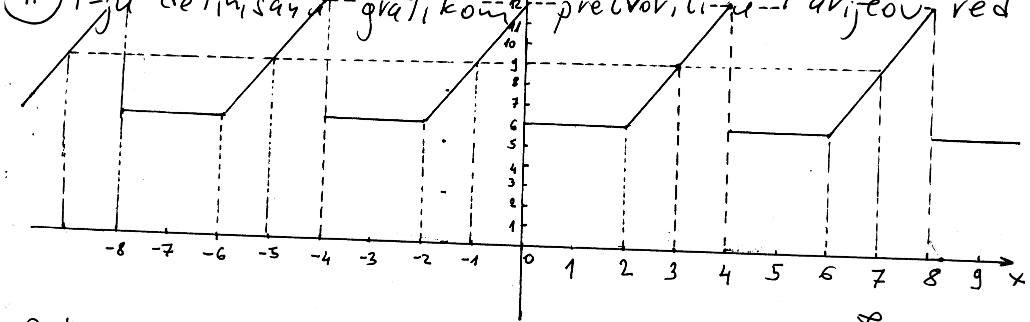
$$+\frac{1}{4k-3}\sin(4k-3)\frac{\pi}{2}+\frac{1}{4k}\underbrace{\sin 4k\frac{\pi}{2}}_{=\sin 2k\pi=0}\Bigg)=$$

$$=\sum_{k=1}^{\infty}\frac{1}{4k-1}\underbrace{\sin\underbrace{(4k-1)\frac{\pi}{2}}_{=3,7,11,15}}_{=-1}+\sum_{k=1}^{\infty}\frac{1}{4k-3}\underbrace{\sin\underbrace{(4k-3)\frac{\pi}{2}}_{=1,5,9,13}}_{=1}$$

$$=\sum_{k=1}^{\infty}\left(-\frac{1}{4k-1}+\frac{1}{4k-3}\right)=\sum_{k=1}^{\infty}\frac{4k-1-4k+3}{(4k-1)(4k-3)}=\sum_{k=1}^{\infty}\frac{2}{(4k-1)(4k-3)}$$

Kako je $f(\frac{\pi}{2})=\frac{\pi}{4}$ to je $\sum_{n=1}^{\infty}\frac{2}{(4n-1)(4n-3)}=\frac{\pi}{4}$.

(#) F-ja definisana grafikom pretvoriti u Furijeov red



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}$.

Rj. Prvo primjetimo da je data f-ja periodična perioda 4, što znači da je možemo pretvoriti u Furijeov red i to dovoljno ju je pretvoriti u Furijeov red na intervalu $(0,4)$.
Data f-ja na intervalu $(0,4)$ je definisana na sljedeći način $f(x)=\begin{cases}6, & x\in(0,2]\\ 3x, & x\in(2,4)\end{cases}$.

Furijeov red na proizvoljnom intervalu $[a,b]$ izgleda

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right)$$

a Furijeovi koeficijenti računaju po formulama

$$a_0=\frac{2}{b-a}\int_a^b f(x)\,dx,\qquad a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx,\quad n=1,2,\dots$$

$$b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx,\quad n=1,2,\dots$$

Što znači Furijeov red na intervalu $[0,4)$ je

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{2}+b_n\sin\frac{n\pi x}{2}\right)$$

Izračunajmo Furijeove koeficijente

$$a_0=\frac12\int_0^4 f(x)\,dx=\frac12\int_0^2 6\,dx+\frac12\int_2^4 3x\,dx=3x\Big|_0^2+\frac32\cdot\frac12\overbrace{x^2\Big|_2^4}^{16-4}=6+\frac{3}{\cancel{4}_1}\cancel{12}^{3}=15$$

$$a_n=\frac12\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx=\frac12\int_0^2 6\cos\frac{n\pi x}{2}\,dx+\frac12\int_2^4 3x\cos\frac{n\pi x}{2}\,dx=\left|\begin{matrix}u=x & dv=\cos\frac{n\pi x}{2}dx\\ du=dx & v=\frac{2}{n\pi}\sin\frac{n\pi x}{2}\end{matrix}\right|$$

$$=\underbrace{3\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^2}_{0-0}+\frac32\left[\underbrace{\frac{2}{n\pi}x\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}-\frac{2}{n\pi}\int_2^4\sin\frac{n\pi x}{2}\,dx\right]=$$

$$=-\frac{3}{n\pi}\int_2^4\sin\frac{n\pi x}{2}\,dx=\frac{3}{n\pi}\cdot\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_2^4=\frac{6}{n^2\pi^2}(1-\cos n\pi),\quad n\neq 0$$

Odavde vidimo $a_n=\begin{cases}0, & n \text{ parno}\\ \frac{12}{n^2\pi^2}, & n \text{ neparno}\end{cases}$ $n\in\mathbb{N}$

$$b_n=\frac12\int_0^4 f(x)\sin\frac{n\pi x}{2}\,dx=\frac12\int_0^2 6\sin\frac{n\pi x}{2}\,dx+\frac12\int_2^4 3x\sin\frac{n\pi x}{2}\,dx=\left|\begin{matrix}u=x & dv=\sin\frac{n\pi x}{2}dx\\ du=dx & v=-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\end{matrix}\right|$$

$$=3\left(-\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}\Big|_0^2+\frac32\left[-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\Big|_2^4+\frac{2}{n\pi}\int_2^4\cos\frac{n\pi x}{2}\,dx\right]=$$

$$=\left(-\frac{6}{n\pi}\right)(\cos n\pi-1)+\frac32\left[\left(-\frac{2}{n\pi}\right)(4\cos 2n\pi-2\cos n\pi)+\underbrace{\frac{2}{n\pi}\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_2^4}_{0-0}\right]$$

$$=\frac{6}{n\pi}(1-\cos n\pi)-\frac{3}{n\pi}(4-2\cos n\pi)=\frac{6}{n\pi}(1-\cos n\pi)-\frac{6}{n\pi}(2-\cos n\pi)$$

$$=\frac{6}{n\pi}(1-\cos n\pi-2+\cos n\pi)=-\frac{6}{n\pi}$$

Prema tome $f(x)\sim\frac{15}{2}+\sum_{n=1}^{\infty}\left(\frac{6}{n^2\pi^2}(1-\cos n\pi)\cos\frac{n\pi x}{2}+\left(-\frac{6}{n\pi}\right)\sin\frac{n\pi x}{2}\right)$

$$=\frac{15}{2}+\sum_{k=1}^{\infty}\left(\frac{12}{(2k-1)^2\pi^2}\cos\frac{(2k-1)\pi x}{2}-\frac{6}{k\pi}\sin\frac{k\pi x}{2}\right)$$

$$f(x)\sim\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\frac{\pi}{2}x}{(2k-1)^2}-\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\frac{\pi}{2}x}{k}$$

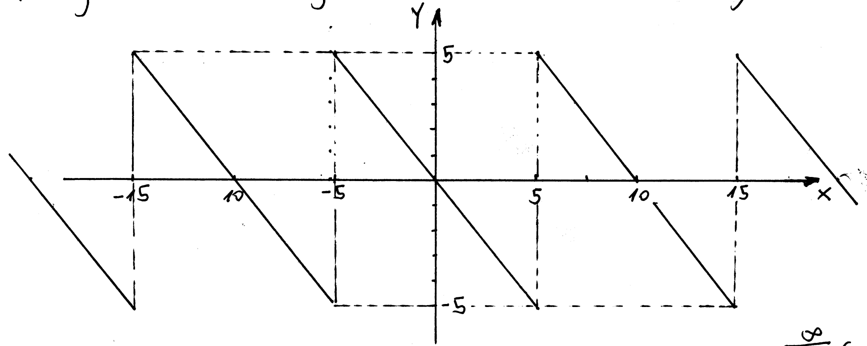
Za $x=2$ imamo

$$f(2)=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi}{(2k-1)^2}-\underbrace{\frac{6}{\pi}\sum_{k=1}^{\infty}\frac{\sin k\pi}{k}}_{=0}$$

$$6=\frac{15}{2}+\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)}{(2k-1)^2}\Rightarrow-\frac{12}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=-\frac32$$

$$\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\left(-\frac32\right)\left(-\frac{\pi^2}{12}\right)=\frac{\pi^2}{2\cdot 4}\qquad\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}=\frac{\pi^2}{8}$$ tražena suma

(#) Funkciju definisanu grafikom pretvoriti u Furijeov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin\frac{n\pi}{50}$.

Rj. Primjetimo da je data f-ja periodična perioda 10. Prema tome dovoljno ju je pretvoriti u Furijeov red na proizvoljnom intervalu perioda 10. Pa posmatrajmo npr. interval $[-5,5]$. F-ja na ovom intervalu ima oblik $f(x) = -x$.

Furijeov red f-je $f(x)$ na intervalu $[a,b]$ ima oblik:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

gdje su $a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$, $a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$ i $b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$ $\quad n=1,2,\dots$

Furijeovi koeficijenti. U našem slučaju interval $[a,b]$ je $[-5,5]$ pa je $b-a = 5+5 = 10$, $\frac{2}{10} = \frac{1}{5}$, $\frac{2n\pi x}{b-a} = \frac{2n\pi x}{10} = \frac{n\pi x}{5}$.

$$a_0 = \frac{1}{5}\int_{-5}^{5}(-x)\,dx = \frac{1}{5}(-1)\cdot\frac{1}{2}x^2\Big|_{-5}^{5} = 0 \qquad d\left(\frac{n\pi x}{5}\right) = \frac{n\pi}{5}dx$$

$$a_n = \frac{1}{5}\int_{-5}^{5}(-x)\cos\frac{n\pi x}{5}\,dx = \begin{vmatrix} u = x & dv = \cos\frac{n\pi x}{5}dx \\ du = dx & v = \frac{5}{n\pi}\sin\frac{n\pi x}{5}\end{vmatrix} =$$

$$= -\frac{1}{5}\left(\overbrace{\frac{5}{n\pi}x\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}^{=0} - \frac{5}{n\pi}\int_{-5}^{5}\sin\frac{n\pi x}{5}\,dx\right) = \frac{1}{n\pi}\left(-\frac{5}{n\pi}\right)\cos\frac{n\pi x}{5}\Big|_{-5}^{5} = 0$$

$$b_n = \frac{1}{5}\int_{-5}^{5}(-x)\sin\frac{n\pi x}{5}\,dx = \begin{vmatrix} u = x & dv = \sin\frac{n\pi x}{5}dx \\ du = dx & v = \frac{5}{n\pi}\left(-\cos\frac{n\pi x}{5}\right)\end{vmatrix} =$$

$$= -\frac{1}{5}\left(\frac{-5}{n\pi}x\cos\frac{n\pi x}{5}\Big|_{-5}^{5} + \frac{5}{n\pi}\int_{-5}^{5}\cos\frac{n\pi x}{5}\,dx\right) =$$

$$= \frac{1}{n\pi}\left(5\cos n\pi - (-5)\cos n\pi\right) - \frac{1}{n\pi}\cdot\underbrace{\frac{5}{n\pi}\sin\frac{n\pi x}{5}\Big|_{-5}^{5}}_{=0} =$$

$$= \frac{10}{n\pi}\cos n\pi = \frac{10}{n\pi}(-1)^n$$

Prema tome

$$-x \sim \sum_{n=1}^{\infty}\frac{10}{n\pi}(-1)^n\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

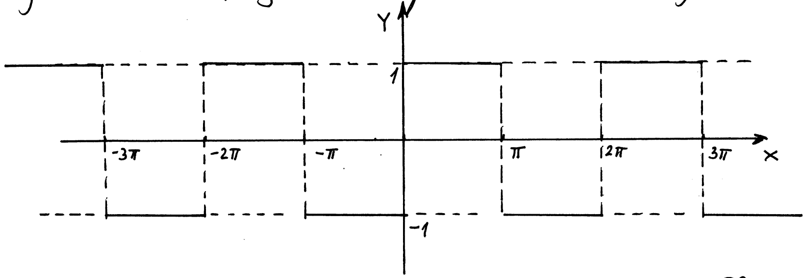
tj. $$-x = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{5} \quad \text{za } \forall x\in[-5,5]$$

Ako za x uzmemo $x = \frac{1}{10}$ imamo:

$$-\frac{1}{10} = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi\cdot\frac{1}{10}}{5}$$

tj. $$\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{50} = -\frac{\pi}{100} \quad \text{tražena suma}$$

F-ju definisanu grafikom pretvoriti u Furijer-ov red.



Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.

Rj. Primjetimo da je data f-ja periodična, period 2π, pa je možemo pretvoriti u Furijer-ov red. Kada je x-osa data u radijanima Furijer-ov red izgleda $\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)$

gdje se Furijer-ovi koeficijenti računaju u obliku (za 2π per. f-ju)

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx+\frac{1}{2}\int_{0}^{\pi}1\,dx=\left(-\frac{1}{\pi}\right)\overbrace{x\Big|_{-\pi}^{0}}^{0-(-\pi)}+\frac{1}{\pi}x\Big|_{0}^{\pi}=-1+1=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx=\left(-\frac{1}{\pi}\right)\frac{1}{n}\sin nx\Big|_{-\pi}^{0}+\frac{1}{\pi}\cdot\frac{1}{n}\sin nx\Big|_{0}^{\pi}=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} b_n\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx=\left(-\frac{1}{\pi}\right)\left(-\frac{1}{n}\right)\cos nx\Big|_{-\pi}^{0}+\frac{1}{\pi}\left(-\frac{1}{n}\right)\cos nx\Big|_{0}^{\pi}$$

$$=\frac{1}{n\pi}(1-\cos n\pi)-\frac{1}{n\pi}(\cos n\pi-1)=\frac{2}{n\pi}(1-\cos n\pi)$$

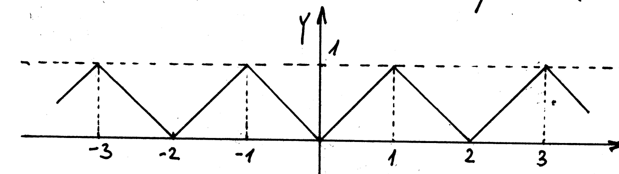
$$1-\cos n\pi=1-(-1)^n=\begin{cases}0, & n=2k\\ 2, & n=2k+1\end{cases}\qquad k=0,1,2,\ldots$$

$$\boxed{\sin(2k+1)\frac{\pi}{2}=(-1)^k}$$

$f(x)\sim\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}$ traženi Furijerov red

$f(\frac{\pi}{2})=1$ $\quad\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\frac{\pi}{2}}{2k+1}=1 \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}$ tražena suma

F-ju definisanu grafikom razviti u Fourierov red. Dobijeni rezultat iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$.



Rj. Sa grafika možemo primjetiti da je f-ja parna i periodična perioda 2. F-ju je dovoljno razviti u Fourierov red u intervalu $[-1,1]$, pa kako je f-ja parna imaćemo da su $b_n=0\ \forall n$.

Ako f-ju označim sa $f(x)$ na intervalu $[-1,1]$ imamo $f(x)=\begin{cases}x, & 0\le x\le 1\\ -x, & -1\le x\le 0\end{cases}$

Ako je $f(x)$ integrabilna f-ja na intervalu $[-l,l]$ Fourierove koeficijente računamo po formulama

$$a_0=\frac{1}{l}\int_{-l}^{l} f(x)\,dx,\quad a_n=\frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx,\quad b_n=\frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Fourierov red f-je $f(x)$ je tad oblika:

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)$$

F-ja je parna:

$$a_0=\frac{2}{l}\int_{0}^{l} f(x)\,dx=2\int_{0}^{1}x\,dx=2\cdot\frac{1}{2}x^2\Big|_{0}^{1}=1$$

$$a_n=\frac{2}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx=2\int_{0}^{1}x\cos n\pi x\,dx=\left|\begin{matrix}u=x & dv=\cos n\pi x\,dx\\ du=dx & v=\frac{1}{n\pi}\sin n\pi x\end{matrix}\right|=$$

$$\underbrace{\frac{2}{n\pi}x\sin n\pi x\Big|_{0}^{1}}_{=0}-\frac{2}{n\pi}\int_{0}^{1}\sin n\pi x\,dx=-\frac{2}{n\pi}\cdot\frac{(-1)}{n\pi}\cos n\pi x\Big|_{0}^{1}=2\frac{\cos n\pi-\cos 0}{n^2\pi^2}$$

$$a_n=2\frac{(-1)^n-1}{n^2\pi^2},\quad b_n=0\ \forall n \Rightarrow f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{\pi^2}\cdot\frac{-4}{(2n-1)^2}\cos(2n+1)\pi x$$

$f(x)=\frac{1}{2}-\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n+1)^2}$ razlaganje f-je u Fourierov red

Za $x=0$, $f(0)=0 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

Razvijanje f-je u red sinusa ili kosinusa u intervalu $[0, l]$

Neka je data f-ja $f(x)$ na intervalu $[0, l]$, $l>0$. Tada možemo definisati novu f-ju $F(x)$ na intervalu $[-l, l]$ koja se podudara sa f-jom $f(x)$ na $[0, l]$ i koja je na $[-l, l]$ parna ili neparna. F-ju $F(x)$ zovemo tada parno ili neparno produženje f-je $f(x)$.

Parno produženje f-je $f(x)$:

$$F(x) = \begin{cases} f(x), & x \in [0, l] \\ f(-x), & x \in [0, l) \end{cases}$$

Znamo od ranije da će tada biti $b_n = 0$.

Neparno produženje f-je $f(x)$:

$$F(x) = \begin{cases} f(x), & x \in [0, l] \\ -f(-x), & x \in [-l, 0) \end{cases}$$

Tada je $a_n = 0$.

Primjer: Data je f-ja $f(x) = \cos x$, za $x \in [0, \pi]$. Uz pomoć ove f-je napraviti f-ju $F(x)$ koja je neparna na $[-\pi, \pi]$.

$$F(x) = \begin{cases} \cos x, & x \in [0, \pi] \\ -\cos x, & x \in [-\pi, 0) \end{cases} \qquad \cos(-x) = \cos x$$

Data je f-ja $f(x) = x^3$ za $x \in [0, 1]$. Uz pomoć ove f-je napraviti f-ju $F(x)$ koja je parna na $[-1, 1]$

$$F(x) = \begin{cases} x^3, & \text{za } x \in [0, 1] \\ -x^3, & \text{za } x \in [-\pi, 0) \end{cases} \qquad (-x)^3 = -x^3$$

Napraviti neparno produženje f-je $f(x) = x^2$ na $[-1, 1]$.

$$F(x) = \begin{cases} x^2, & x \in [0, 1] \\ -x^2, & x \in [-\pi, 0) \end{cases} \qquad (-x)^2 = x^2$$

Razviti f-ju $f(x) = x\left(\frac{\pi}{2} - x\right)$ po sinusima višestrukih uglova u intervalu $\left(0, \frac{\pi}{2}\right)$.

Rj. Furijer-ov red f-je $f(x)$ na intervalu $[a, b]$ je oblika

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2n\pi x}{b-a} + b_n \sin\frac{2n\pi x}{b-a}\right)$$

gdje se Furijeovi koeficijenti računaju po formuli

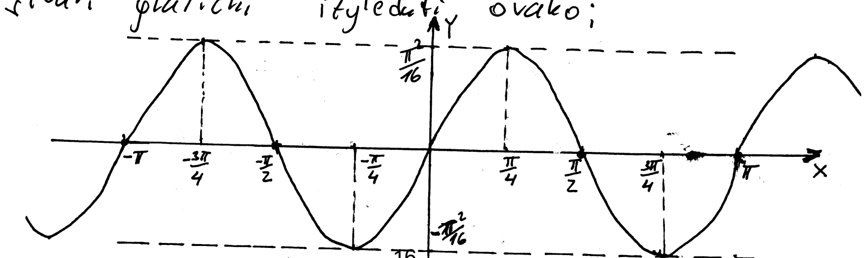
$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx, \quad a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx, \quad b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx.$$

Da bi razvili f-ju po sinusima višestrukih uglova trebamo naštimati da je $a_n = 0$, a prema formuli za a_n, a_n će biti jednak nuli akko je interval $[a, b]$ simetričan u odnosu na 0 i ako je $f(x)$ neparna f-ja.

Prema tome da bi našu f-ju $f(x) = x\left(\frac{\pi}{2} - x\right)$ razvili u red po sinusima, pravimo neparno produženje f-je $f(x)$ (novu f-ju ćemo obilježiti sa $f^*(x)$)

$$f^*(x) = \begin{cases} f(x), & x \in \left(0, \frac{\pi}{2}\right) \\ -f(-x), & x \in \left(-\frac{\pi}{2}, 0\right) \end{cases} = \begin{cases} x\left(\frac{\pi}{2} - x\right), & x \in \left(0, \pi/2\right) \\ x\left(\frac{\pi}{2} + x\right), & x \in \left(-\frac{\pi}{2}, 0\right). \end{cases}$$

Primjetimo da f-ja $f^*(x)$ koju pretvaramo u Furijerov red će u stvari grafički izgledati ovako:



Izračunajmo Furijeove koeficijente. Posmatramo interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

$b-a=\frac{\pi}{2}-(-\frac{\pi}{2})=\pi$, $\quad \frac{2}{b-a}=\frac{2}{\pi}$, $\quad \frac{2n\pi x}{b-a}=2nx$

$$b_n=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\underbrace{\underset{\text{neparna f-ja}}{f^*(x)}\,\underset{\text{neparna f-ja}}{\sin 2nx}}_{\text{parna f-ja}}\,dx = \frac{4}{\pi}\int_0^{\pi/2} f^*(x)\sin\sin 2nx\,dx = \frac{4}{\pi}\int_0^{\pi/2} x\left(\frac{\pi}{2}-x\right)\sin 2nx\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi/2}\left(\frac{\pi}{2}x-x^2\right)\sin 2nx\,dx = \frac{4}{\pi}\cdot\frac{\pi}{2}\int_0^{\pi/2} x\sin 2nx\,dx - \frac{4}{\pi}\int_0^{\pi/2} x^2\sin 2nx\,dx \overset{(*)}{=}$$

$$I_1 = 2\int_0^{\pi/2} x\sin 2nx\,dx = \left|\begin{matrix} u=x & dv=\sin 2nx \\ du=dx & v=\frac{-1}{2n}\cos 2nx\end{matrix}\right| = -\frac{2x}{2n}\cos 2nx\Big|_0^{\frac{\pi}{2}} + 2\cdot\frac{1}{2n}\int_0^{\pi/2}\cos 2nx\,dx$$

$$= \left(-\frac{\pi}{2n}\cos\pi nx - 0\right) + \frac{1}{n}\cdot\frac{1}{2n}\sin 2nx\Big|_0^{\frac{\pi}{2}} = \frac{(-1)^{n+1}\pi}{2n}$$

$$I_2 = \frac{4}{\pi}\int_0^{\pi/2} x^2\sin 2nx\,dx = \left|\begin{matrix} u=x^2 & dv=\sin 2nx\,dx \\ du=2x\,dx & v=\frac{-1}{2n}\cos 2nx\,dx\end{matrix}\right| = \frac{4}{\pi}\cdot\frac{(-1)}{2n}x^2\cos 2nx\Big|_0^{\pi/2}$$

$$+\frac{1}{2n}\cdot 2\cdot\frac{4}{\pi}\int_0^{\pi/2} x\cos 2nx\,dx = \left|\begin{matrix} u=x & dv=\cos 2nx\,dx \\ du=dx & v=\frac{1}{2n}\sin 2nx\end{matrix}\right| = \frac{-2}{n\pi}\left(\overset{\frac{\pi^2}{4}}{\left(\frac{\pi}{2}\right)^2}(-1)^n - 0\right)$$

$$+\frac{4}{n\pi}\left[\underbrace{\frac{1}{2n}x\sin 2nx\Big|_0^{\frac{\pi}{2}}}_{=0} - \frac{1}{2n}\int_0^{\pi/2}\sin 2nx\right] = \frac{(-1)^{n+1}\pi}{2n} + \frac{4}{n\pi}\cdot\frac{1}{4n^2}\cos 2n\pi\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{(-1)^{n+1}\pi}{2n} + \frac{1}{n^3\pi}\left((-1)^n - 1\right)$$

$$\overset{(*)}{=} \frac{(-1)^{n+1}\pi}{2n} - \frac{(-1)^{n+1}\pi}{2n} - \frac{(-1)^n-1}{n^3\pi} = \frac{1-(-1)^n}{n^3\pi}$$ vrijednost koeficijenta b_n

$$f(x)\sim\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3\pi}\sin 2nx = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin 2(2k-1)x}{(2k-1)^3}$$ traženi Furijeov red

za $x\in(0,\frac{\pi}{2})$.

Razviti u intervalu $(0,\pi)$ po sinusima višestrukih lukova f-ju $f(x)=\frac{\pi}{4}$. Dobijeni razvoj upotrebite za sumiranje redova brojeva

a) $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$; $\quad$ b) $1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\dots$;

c) $1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\dots$.

Rj. Pravimo neparno produženje f-je $f(x)$, $\quad \overline{f(x)}=\begin{cases}\frac{\pi}{4}, & x\in(0,\pi)\\ -\frac{\pi}{4}, & x\in(-\pi,0)\end{cases}$

$\overline{f(x)}$ neparna, $a_n=0$, $n=0,1,2,\dots$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin nx\,dx = \frac{2}{\pi}\cdot\frac{\pi}{4}\int_0^{\pi}\sin nx\,dx = \frac{1}{2}\cdot\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi}$$

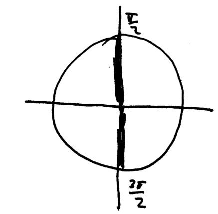
$$=-\frac{1}{2n}(\cos n\pi - \cos 0^\circ) = -\frac{1}{2n}\left((-1)^n-1\right),$$ za $n=2k$, $b_n=0$, $k=1,2,\dots$; za $n=2k+1$, $b_n=\frac{1}{n}$, $k=0,1,2,\dots$

$$f(x)=\sum_{n=1}^{\infty} b_n\sin nx$$

$$\overline{f(x)}=\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1} \quad \text{tj.} \quad \frac{\pi}{4}=\sum_{k=0}^{\infty}\frac{\sin(2k+1)x}{2k+1}, \quad x\in(0,\pi)$$

a) za $x=\frac{\pi}{2}$ imamo $\frac{\pi}{4}=\sum_{k=0}^{\infty}\frac{\sin\frac{(2k+1)\pi}{2}}{2k+1}$,

$k=0$, $2k+1=1$, $\sin\frac{\pi}{2}=1$
$k=1$, $2k+1=3$, $\sin\frac{3\pi}{2}=-1$
$k=2$, $2k+1=5$, $\sin\frac{5\pi}{2}=1$



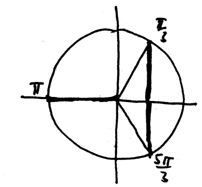
$$\sin\frac{(2k+1)\pi}{2}=\begin{cases}1, & k=0,2,4,\dots\\ -1, & k=1,3,5,\dots\end{cases}$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots=\frac{\pi}{4}$$

$$\sin\frac{(2k+1)\pi}{3}=\begin{cases}\frac{\sqrt{3}}{2}, & k=0,3,6,\dots\\ 0, & k=1,4,7,\dots\\ -\frac{\sqrt{3}}{2}, & k=2,5,8,\dots\end{cases}$$

b) za $x=\frac{\pi}{3}$ imamo $\frac{\pi}{3}=\sum_{k=0}^{\infty}\frac{\sin\frac{(2k+1)\pi}{3}}{2k+1}$

$k=0$, $2k+1=1$, $\sin\frac{\pi}{3}$
$k=1$, $2k+1=3$, $\sin\pi$
$k=2$, $2k+1=5$, $\sin\frac{5\pi}{3}$
$k=3$, $2k+1=7$, $\sin\frac{7\pi}{3}$



$$\frac{\sqrt{3}}{2}\left(1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\dots\right)=\frac{\pi}{4}$$

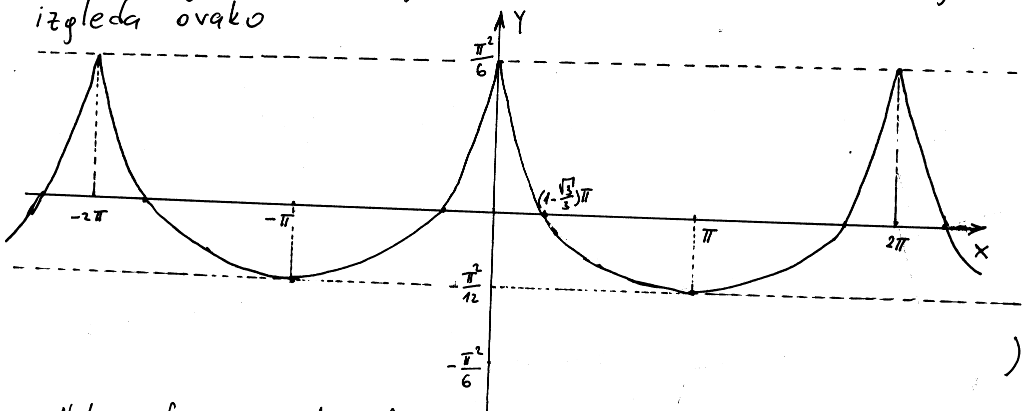
$$\frac{\pi}{2\sqrt{3}}=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\dots$$

c) za vježbu

uputa: za x se uzme $\frac{\pi}{4}$ $\qquad$ rješenje: $\frac{\pi}{2\sqrt{2}}=1+\frac{1}{3}-\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+\frac{1}{11}-\dots$

Razviti f-ju $f(x)=\dfrac{3x^2-6\pi x+2\pi^2}{12}$ u red po kosinusima u intervalu $(0,\pi)$.

Rj. F-ju koju razvijamo u Furijeov red po kosinusima grafički izgleda ovako



Neka je $f(x)$ 2π periodična f-ja.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \quad \text{Furijeov red f-je } f(x)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx,\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx \quad \text{Furijeovi koeficijenti f-je } f(x)$$

Ako je $f(x)$ parna tada je $f(x)\sin nx$ neparna $\Rightarrow b_n=0 \ \forall n\in\mathbb{Z}$

Ako je $f(x)$ neparna tada je $f(x)\cos nx$ neparna $\Rightarrow a_n=0 \ \forall n\in\mathbb{Z}$

Trebamo napraviti parno produženje f-je $f(x)$ (novu f-ju nazovimo $f^*(x)$)

$$f^*(x)=\begin{cases} f(x), & x\in(0,\pi)\\ f(-x), & x\in(-\pi,0)\end{cases} = \begin{cases} \frac{1}{12}(3x^2-6\pi x+2\pi^2), & x\in(0,\pi)\\ \frac{1}{12}(3x^2+6\pi x+2\pi), & x\in(-\pi,0)\end{cases}$$

Izračunajmo Fourijeove koeficijente

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f^*(x)\,dx \overset{f^*\text{ parna}}{=} \frac{2}{\pi}\int_0^{\pi}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2-6\pi x+2\pi^2)\,dx=\frac{1}{6\pi}\left(3\cdot\frac{1}{3}x^3\Big|_0^{\pi}-6\pi\cdot\frac{1}{2}x^2\Big|_0^{\pi}+2\pi^2\cdot x\Big|_0^{\pi}\right)=\frac{1}{6\pi}(\pi^3-3\pi^3+2\pi^3)=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f^*(x)\cos nx\,dx \overset{f^*(x)\text{ parna}}{=} \frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{1}{12}(3x^2-6\pi x+2\pi^2)\cos nx\,dx=$$

$$=\frac{1}{6\pi}\left(3\int_0^{\pi}x^2\cos nx\,dx-6\pi\int_0^{\pi}x\cos nx\,dx+2\pi^2\int_0^{\pi}\cos nx\,dx\right)\overset{(*)}{=}$$

$$I_1=\int_0^{\pi}x^2\cos nx\,dx=\left|\begin{matrix}u=x^2 & dv=\cos nx\,dx\\ du=2x\,dx & v=\frac{1}{n}\sin nx\end{matrix}\right|=\frac{1}{n}x^2\sin nx\Big|_0^{\pi}-\frac{2}{n}\int_0^{\pi}x\sin nx\,dx=$$

$$=\left|\begin{matrix}u=x & dv=\sin nx\,dx\\ du=dx & v=-\frac{1}{n}\cos nx\end{matrix}\right|=\frac{1}{n}\underbrace{(\pi^2\sin n\pi-0)}_{=0}-\frac{2}{n}\left(-\frac{1}{n}x\cos nx\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)$$

$$=\frac{2}{n^2}(\pi\cos n\pi-0)-\frac{2}{n^2}\sin nx\Big|_0^{\pi}=(-1)^n\frac{2\pi}{n^2}$$

$$I_2=\int_0^{\pi}x\cos nx\,dx=\left|\begin{matrix}u=x & dv=\cos nx\,dx\\ du=dx & v=\frac{1}{n}\sin nx\end{matrix}\right|=\frac{1}{n}x\sin nx\Big|_0^{\pi}-\frac{1}{n}\int_0^{\pi}\sin nx\,dx=$$

$$=\frac{1}{n}\underbrace{(\pi\sin n\pi-0)}_{=0}-\frac{1}{n}\left(-\frac{1}{n}\right)\cos nx\Big|_0^{\pi}=\frac{1}{n^2}(\cos n\pi-\cos 0)=\frac{1}{n^2}((-1)^n-1)$$

$$I_3=\int_0^{\pi}\cos nx\,dx=\frac{1}{n}\sin nx\Big|_0^{\pi}=\frac{1}{n}(\sin n\pi-\sin 0)=0$$

$$\overset{(*)}{=}\frac{1}{\cancel{2\pi}}(-1)^n\frac{\cancel{2\pi}}{n^2}-\frac{1}{n^2}((-1)^n-1)=\frac{1}{n^2}((-1)^n-(-1)^n+1)=\frac{1}{n^2}$$

$$f(x)=\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx,\quad x\in(0,\pi)$$

razvoj f-je $f(x)$ u red po kosinusima

(Primjetimo da dobijeni rezultat možemo iskoristiti za sumiranje reda $\sum_{n=1}^{\infty}\frac{1}{n^2}$. Naime ako stavimo $x=0$ imamo

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$).

F-ju $f(x)=\sin x$ razložiti u red po kosinusima u intervalu $[0,\pi]$.

Rj. Pravimo parno produženje f-je $f(x)$

$$\bar f(x)=\begin{cases}\sin x, & x\in[0,\pi]\\ -\sin x, & x\in[-\pi,0)\end{cases}$$

$\bar f(x)$ je parna f-je na $[-\pi,\pi] \Rightarrow b_n=0 \quad (n=1,2,\dots)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=-\frac{2}{\pi}\cos x\Big|_0^{\pi}=-\frac{2}{\pi}(\cos\pi-\cos 0)=-\frac{2}{\pi}\cdot(-2)=\frac{4}{\pi}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx=\frac{2}{\pi}\cdot\frac{1}{2}\int_0^{\pi}\big(\sin(x+nx)+\sin(x-nx)\big)\,dx$$

$\sin\alpha\cos\beta=\frac{1}{2}\sin(\alpha-\beta)+\frac{1}{2}\sin(\alpha+\beta)$

$\cos\alpha\cos\beta=\frac{1}{2}\cos(\alpha-\beta)+\frac{1}{2}\cos(\alpha+\beta)$

$\sin\alpha\sin\beta=\frac{1}{2}\cos(\alpha-\beta)-\frac{1}{2}\cos(\alpha+\beta)$

$$=\frac{1}{\pi}\int_0^{\pi}\sin(n+1)x\,dx+\frac{1}{\pi}\int_0^{\pi}\sin(1-n)x\,dx$$

$$=-\frac{1}{\pi}\cdot\frac{1}{n+1}\cos(n+1)x\Big|_0^{\pi}-\frac{1}{\pi}\cdot\frac{1}{1-n}\cos(1-n)x\Big|_0^{\pi}=-\frac{1}{\pi(n+1)}\Big(\underbrace{\cos(n+1)\pi}_{(-1)^{n+1}}-\underbrace{\cos 0^\circ}_{1}\Big)+\frac{1}{\pi(n-1)}\Big(\underbrace{\cos(1-n)\pi}_{\cos(n-1)\pi}-\cos 0^\circ\Big)=$$

$$=\frac{1}{\pi}\left[-\frac{1}{n+1}\big((-1)^{n+1}-1\big)+\frac{1}{n-1}\big((-1)^{n-1}-1\big)\right]=\frac{1}{\pi}\big((-1)^{n+1}-1\big)\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$$

$$=\frac{(-1)^{n+1}-1}{\pi}\cdot\frac{\cancel{n}+1-\cancel{n}+1}{(n-1)(n+1)}=\frac{2\left[(-1)^{n+1}-1\right]}{(n^2-1)\pi}\,,\qquad \begin{array}{l} n=2k+1, \quad (-1)^{n+1}-1=0\\ n=2k, \quad (-1)^{n+1}-1=-2\end{array}$$

$$a_{2k}=\frac{-4}{\pi(4k^2-1)}\,,\quad k=1,2,\dots$$

$$\sin x=\frac{2}{\pi}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}\,,\quad x\in[-\pi,\pi]$$

razlaganje f-je $f(x)$ po kosinusima

Zadaci za vježbu

U zadacima 4377—4390 razviti u Furije-ov red date funkcije u datim intervalima.

4377. Funkciju $y=x^2$ u intervalu $(0,\pi)$ u sinusni red.

4378. Funkciju x^3 u intervalu $(-\pi,\pi)$.

4379. Funkciju $f(x)=\begin{cases}1 & \text{za } -\pi<x<0,\\ 3 & \text{za } 0<x<\pi.\end{cases}$

4380. Funkciju $f(x)=\begin{cases}1 & \text{za } 0<x<h\\ 0 & \text{za } h<x<\pi\end{cases}$ u kosinusni red $(0<h<\pi)$.

4381. Neprekidnu funkciju $f(x)=\begin{cases}1 & \text{za } x=0,\\ 0 & \text{za } 2h<x<\pi,\end{cases}$ i linearnu u intervalu $(0,\ 2h)$, — u kosinusni red. $\left(0<h<\frac{\pi}{2}\right)$.

4382. Funkciju $|x|$ u intervalu $(-l,\ l)$.

4383. Funkciju e^x-1 u intervalu $(0,\ 2\pi)$.

4384. Funkciju e^x u intervalu $(-l,\ l)$.

4385. Funkciju $\cos ax$ u intervalu $(-\pi,\pi)$ (a je neceo broj).

4386. Funkciju $\sin ax$ u intervalu $(-\pi,\pi)$ (a je neceo broj).

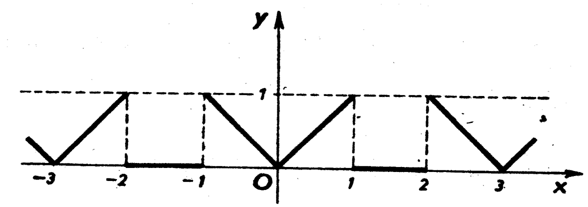
4387. Funkciju $\sin ax$ (a je ceo broj) u intervalu $(0,\pi)$ u kosinusni red.

4388. Funkciju $\cos ax$ (a je ceo broj) u intervalu $(0,\pi)$ u sinusni red.

4389. Funkciju $\mathrm{sh}\,ax$ u intervalu $(-\pi,\pi)$.

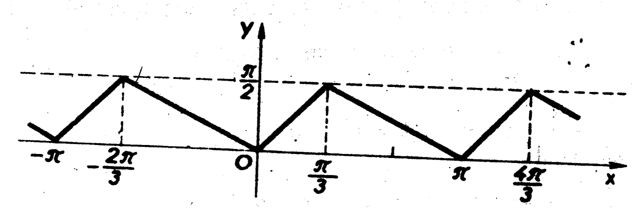
4390. Funkciju $\mathrm{ch}\,x$ u intervalu $(0,\pi)$ u kosinusni red i u sinusni red.

4391. Razviti u Furije-ov red funkciju definisanu grafikom na sl. 74.



Sl. 74

4392*. Razviti u Furije-ov red funkciju definisanu grafički na sl. 75.



Sl. 75

Rješenja

4377. $\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\left\{\frac{\pi^2}{n}+\right.$

$\left.+\frac{2}{n^3}[(-1)^n-1]\right\}\sin nx.$

4378. $\sum_{n=1}^{\infty}(-1)^n\left(\frac{12}{n^2}-\frac{2\pi^2}{n}\right)\sin nx.$

4379. $2+\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)x}{2n+1}.$

4380. $\frac{2h}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{\sin nh}{nh}\cos nx\right].$

4384. $\frac{e^l-e^{-l}}{2l}+l(e^l-e^{-l})\sum_{n=1}^{\infty}\frac{(-1)^n\cos\frac{n\pi x}{l}}{l^2+n^2\pi^2}+$

$+\pi(e^l-e^{-l})\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n\sin\frac{n\pi x}{l}}{l^2+n^2\pi^2}=$

$=\operatorname{sh} l\left[\frac{1}{l}+2\sum_{n=1}^{\infty}(-1)^n\frac{l\operatorname{sos}\frac{n\pi x}{l}-\pi n\sin\frac{n\pi x}{l}}{l^2+n^2\pi^2}\right].$

4385. $\frac{2\sin\pi a}{\pi}\left(\frac{1}{2a}+\frac{a\cos x}{1-a^2}-\frac{a\cos 2x}{2^2-a^2}+\cdots\right).$

4386. $\frac{2\sin\pi a}{\pi}\left(\frac{\sin x}{1-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\cdots\right).$

4387. $\sin ax=\begin{cases}\frac{4a}{\pi}\left[\frac{\cos x}{a^2-1}+\frac{\cos 3x}{a^2-3^2}+\frac{\cos 5x}{a^2-5^2}+\cdots\right] & \text{ako je } a \text{ paran broj.}\\ \frac{4a}{\pi}\left[\frac{1}{2a^2}+\frac{\cos 2x}{a^2-2^2}+\frac{\cos 4x}{a^2-4^2}+\cdots\right] & \text{ako je } a \text{ neparan broj.}\end{cases}$

4388. $\cos ax=\begin{cases}-\frac{4}{\pi}\left[\frac{\sin x}{a^2-1^2}+\frac{3\sin 3x}{a^2-3^2}+\frac{5\sin 5x}{a^2-5^2}+\cdots\right] & \text{ako je } a \text{ paran broj.}\\ -\frac{4}{\pi}\left[\frac{2\sin 2x}{a^2-2^2}+\frac{4\sin 4x}{a^2-4^2}+\frac{6\sin 6x}{a^2-6^2}+\cdots\right] & \text{ako je } a \text{ neparan broj.}\end{cases}$

4381. $\frac{2h}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\sin nh}{nh}\right)^2\cos nx\right].$

4382. $\frac{l}{2}-\frac{4l}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos\left[\frac{(2n+1)\pi x}{l}\right]}{(2n+1)^2}.$

4383. $\frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\cos nx}{1+n^2}-\right.\right.$

$\left.\left.-\frac{n\sin x}{1+n^2}\right)\right]-1)$

4389. $\frac{2\operatorname{sh} a\pi}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{a^2+n^2}\sin nx.$

4390. $\frac{\operatorname{sh}\pi}{\pi}\left[1+2\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{1+n^2}\right];\ \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n\operatorname{ch}\pi}{1+n^2}n\sin nx.$

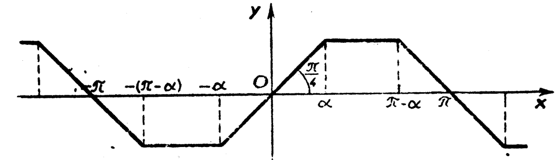
4391. $f(x)=\frac{1}{3}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{\sin\frac{2\pi n}{3}}{n}-\right.$

$\left.-\frac{3\left(1-\cos\frac{2\pi n}{3}\right)}{2\pi n^2}\right]\cos\frac{2\pi nx}{3}=$

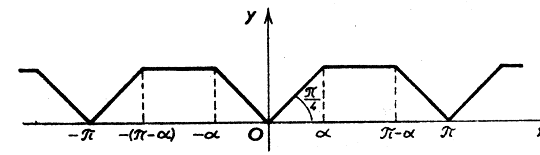
$=\frac{1}{3}+\frac{\sqrt{3}}{\pi}\left(\frac{\cos\frac{2\pi x}{3}}{1}-\frac{\cos\frac{4\pi x}{3}}{2}+\frac{\cos\frac{8\pi x}{3}}{4}-\cdots\right)-$

$-\frac{9}{2\pi^2}\left(\frac{\cos\frac{2\pi x}{3}}{1^2}+\frac{\cos\frac{4\pi x}{3}}{2^2}+\frac{\cos\frac{8\pi x}{3}}{4^2}+\cdots\right).$

4393*. Razviti u Furije-ov red funkcije čiji su grafici prikazani na sl. 76 i 77.



Sl. 76



Sl. 77

4394. Razviti funkciju $x(\pi-x)$ u sinusni red u intervalu $(0, \pi)$, i koristeći dobijeni rezultat naći zbir reda

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots+\frac{(-1)^{n-1}}{(2n-1)^3}+\cdots$$

4395. Data je funkcija $\varphi(x)=(\pi^2-x^2)^2$.

a) Uveriti se da ova funkcija zadovoljava jednakosti:

$$\varphi(-\pi)=\varphi(\pi),\ \varphi'(-\pi)=\varphi'(\pi)\ \text{ i }\ \varphi''(-\pi)=\varphi''(\pi)$$
$$[\text{ali } \varphi'''(-\pi)\neq\varphi'''(\varphi)].$$

b) Koristeći dobijene jednakosti razviti funkciju $\varphi(x)$ u Furije-ov red u intervalu $(-\pi, \pi)$.

c) Izračunati zbir reda

$$1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\cdots+\frac{(-1)^{n-1}}{n^4}+\cdots$$

Rješenja

4392*. $f(x)=\frac{\pi}{6}+\frac{3}{2\pi}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi}{3}}{n^2}\left(\cos\frac{n\pi}{3}\sin 2nx-\sin\frac{nx}{3}\cos 2nx\right)=$

$=\frac{\pi}{6}+\frac{3\sqrt{3}}{8\pi}\left(\frac{\sin 2x}{1^2}-\frac{\sin 4x}{2^2}+\frac{\sin 8x}{4^2}-\frac{\sin 10x}{5^2}+\cdots\right)-$

$-\frac{9}{8\pi}\left(\frac{\cos 2x}{1^2}+\frac{\cos 4x}{2^2}+\frac{\cos 8x}{4^2}+\frac{\cos 10x}{5^2}+\cdots\right).$

Iskoristiti rezultat zadatka 4368.

4393*. 1) $f(x)=\frac{4}{\pi}\left(\frac{\sin\alpha\cdot\sin x}{1^2}+\frac{\sin 3\alpha\cdot\sin 3x}{3^2}+\cdots\right)$

2) $f(x)=\frac{\alpha(\pi-\alpha)}{\pi}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos 2n\alpha}{n^2}\cos 2nx=$

$=\frac{\alpha(\pi-\alpha)}{\pi}-\frac{2}{\pi}\left(\frac{\sin^2\alpha\cdot\cos 2x}{1^2}+\frac{\sin^2 2\alpha\cdot\cos 4x}{2^2}+\cdots\right).$

Iskoristiti rezultat zadatka 4371.

4394. $\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)^3};\ \frac{\pi^3}{32}.$

4395. $\frac{8}{15}\pi^4-48\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^4};$ c) $\frac{7}{720}\pi^4.$

Funkcija dvije nezavisne promjenjive

Neka je S neprazan podskup prostora $\mathbb{R}^2$ i $T \subseteq \mathbb{R}$. Ako svakoj tački $M(x,y) \in S$ možemo unaprijed po datom pravilu f pridružiti jednu i samo jednu realnu vrijednost $z \in T$, tada kažemo da je data realna f-ja dvije realne promjenjive f iz $\mathbb{R}^2$ u $\mathbb{R}$ (sa skupa $S \subseteq \mathbb{R}$ u skup $T \subseteq \mathbb{R}$) i pišemo $z = f(x,y)$. Skup S na kojem je određena f-ja f naziva se domen ili definiciono područje f-je f (označavat ćemo ga sa $\mathcal{D}(f)$), a skup $f(A)$ skup vrijednosti f-je f ili kodomen (označavat ćemo ga sa $\mathcal{R}(f)$). Ako za f-ju, zadanu analitički (formulom) nije data oblast njene definisanosti, onda se pod njom podrazumjeva skup svih tačaka $M \in \mathbb{R}^2$ u kojoj f-ja, odnosno njen analitički izraz imaju određenu realnu vrijednost.

Za svaku od sljedećih f-ja, izračunati $f(3,2)$, i odrediti i skicirati domen.

a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ b) $f(x,y) = x \ln(y^2 - x)$

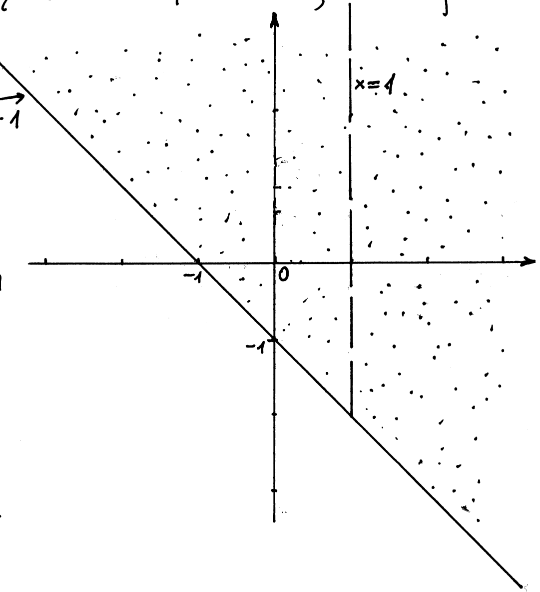
Rj.

a) $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$

Izraz za f-ju $f(x,y)$ ima smisla ako je nazivnik različit od nule i ako je vrijednost pod korijenom nenegativna:

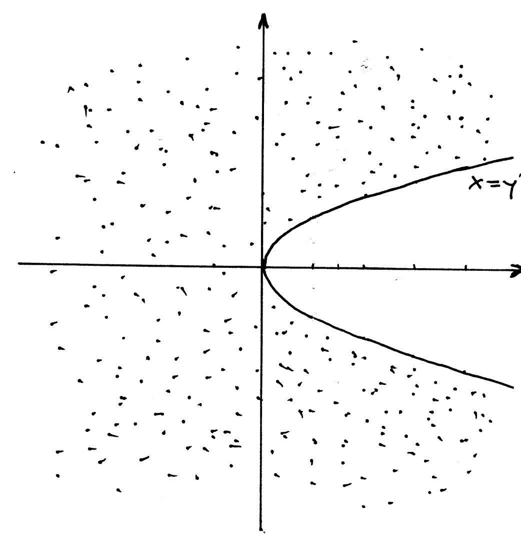
$$\begin{array}{l} x-1 \neq 0 \\ x+y+1 \geq 0 \end{array} \Rightarrow \begin{array}{l} x \neq 1 \\ x+y \geq -1 \end{array}$$

Domen f-je f je $D = \{(x,y) \in \mathbb{R}^2 \mid x+y \geq -1,\ x \neq 1\}$



b) $f(3,2) = 3\ln(2^2-3)$
$= 3\ln(4-3) = 3\underbrace{\ln 1}_{=0} =$
$= 0$

Izraz $\ln(y^2 - x)$ je definisan samo ako je $y^2 - x > 0$



$D = \{(x,y) \mid x < y^2\}$

Odrediti domen i rang f-je $g(x,y)=\sqrt{9-x^2-y^2}$

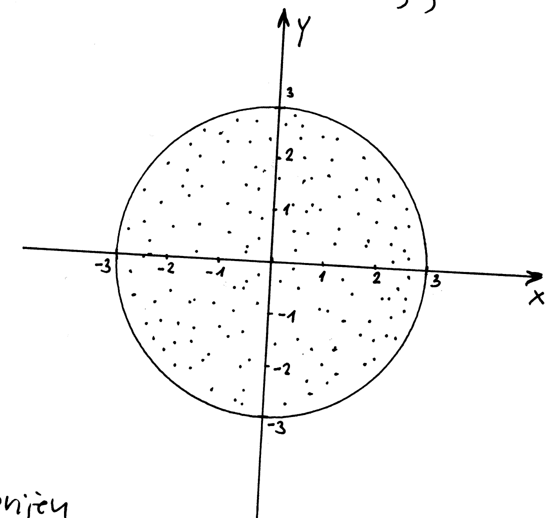
Rj.

F-ja ima smisla akko $9-x^2-y^2 \geq 0$

$$x^2+y^2 \leq 9$$

Domen f-je $g(x,y)$ je $D=\{(x,y)\in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$

(znamo da je $x^2+y^2=9$ krug sa centrom u tački $C(0,0)$ poluprečnika $r=3$).



Rang f-je g je

$\{z\in \mathbb{R} \mid z=\sqrt{9-x^2-y^2},\ (x,y)\in D\}$

Primjetimo da je

$$9-x^2-y^2 \leq 9 \quad \text{za} \quad \forall (x,y)\in D$$

pa je $\sqrt{9-x^2-y^2} \leq 3$

z je pozitivan kvadratni korijen

$$z \geq 0$$

Prema tome, rang f-je $g(x,y)$ je

$$\{z \mid 0 \leq z \leq 3\} = [0,3]$$

Skicirati graf f-je $f(x,y)=6-3x-2y$.

Rj.

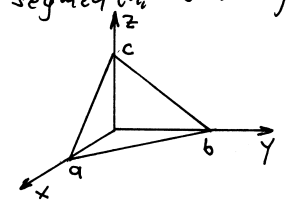
Graf f-je $f(x,y)$ ima jednačinu $z=6-3x-2y$

$$3x+2y+z=6$$

ovo predstavlja ravan.

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

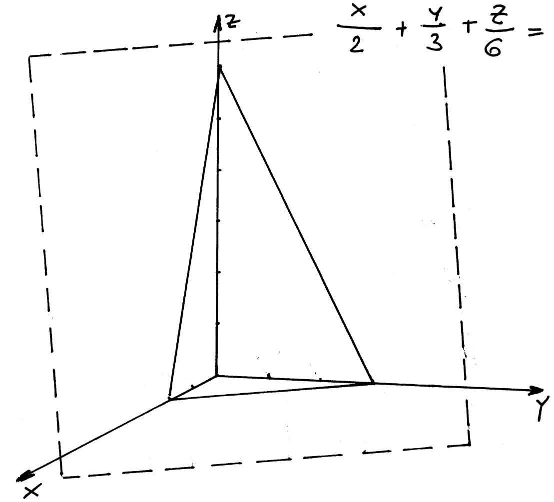
segmentni oblik jednačine ravni



U našem slučaju

$$3x+2y+z=6 \quad /:6$$

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$$



Skicirati graf f-je $g(x,y)=\sqrt{9-x^2-y^2}$.

Rj. Graf f-je ima jednačinu $z=\sqrt{9-x^2-y^2}$

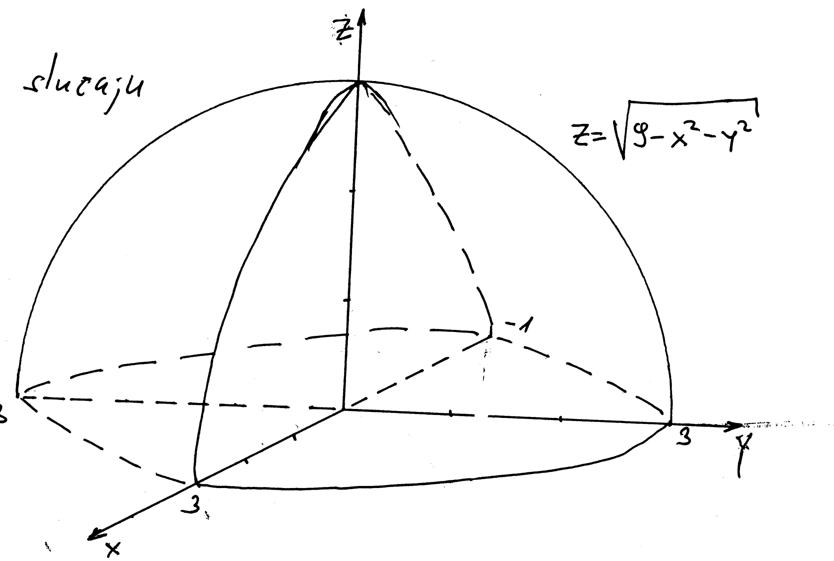
$$z=\sqrt{9-x^2-y^2} \quad /^2$$

$$z^2=9-x^2-y^2$$

$$x^2+y^2+z^2=9$$

$x^2+y^2+z^2=R^2$ je jednačina sfere sa centrom u koordinatnom početku poluprečnika R

U našem slučaju



Limesi i neprekidnost

Definicija Neka je f f-ja dvije varijable čiji je domen D koji sadrži tačku (a,b) i okolinu tačke (a,b). Tada kažemo da je L limes f-je $f(x,y)$ kad se (x,y) približava (ili teži) tački (a,b) i pišemo

$$\lim_{(x,y)\to(a,b)} f(x,y)=L$$

ako za svaki realan $\varepsilon>0$ postoji odgovarajući broj $\delta>0$ takav da

ako $(x,y)\in D$ i $0<\sqrt{(x-a)^2+(y-b)^2}<\delta$ tada $|f(x,y)-L|<\varepsilon$.

Tvrdnja 1

Ako $f(x,y)\to L_1$ kad $(x,y)\to(a,b)$ duž puta C_1 i $f(x,y)\to L_2$ kad $(x,y)\to(a,b)$ duž puta C_2, gdje je $L_1\neq L_2$, tada $\lim_{(x,y)\to(a,b)} f(x,y)$ ne postoji.

(#) Pokazati da $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ ne postoji.

Rj.

Neka je $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$. Približavajmo se tački $(0,0)$ duž x-ose.

Tada je $y=0$ pa imamo

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,0)\to(0,0)} f(x,0) = \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} = 1 \quad \text{za } \forall x\neq 0$$

Sada se želimo približavati tački $(0,0)$ duž y-ose. Tj. $x=0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(0,y)\to(0,0)} f(0,y) = \lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} = -1 \quad \text{za } \forall y\neq 0$$

Kako limes ima dvije različite vrijednosti duž dvije različite linije dati limes ne postoji.

(#) Ako je $f(x,y)=\dfrac{xy}{x^2+y^2}$ da li postoji $\lim\limits_{(x,y)\to(0,0)} f(x,y)$?

Rj.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left|\begin{array}{l}\text{približavajmo se}\\ \text{tački } (0,0) \text{ duž}\\ \text{prave } x=y\end{array}\right| = \lim_{(x,x)\to(0,0)} f(x,x) = \lim_{(x,x)\to(0,0)} \frac{x^2}{2x^2} = \frac{1}{2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left|\begin{array}{l}\text{približavajmo se}\\ \text{tački } (0,0) \text{ duž}\\ \text{prave } y=0\\ \text{(duž x-ose)}\end{array}\right| = \lim_{(x,0)\to(0,0)} f(x,0) = \lim_{(x,0)\to(0,0)} \frac{0}{x^2} = 0$$

Kako smo dobili dvije različite vrijednosti za limes, duž dva različita puta, dati limes ne postoji.

Ako je $f(x,y)=\frac{xy^2}{x^2+y^4}$ da li $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ postoji?

Rj.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left| \begin{array}{l} \text{približavajmo se} \\ \text{tački } (0,0) \text{ duž} \\ \text{prave } y=mx \\ \text{gdje je } m \text{ koeficijent} \\ \text{pravca} \end{array} \right| = \lim_{(x,mx)\to(0,0)} f(x,mx) = \lim_{(x,mx)\to(0,0)} \frac{m^2x^3 \;/ :x^2}{x^2+m^4x^4 \;/ :x^2}$$

$$= \lim_{(x,mx)\to(0,0)} \frac{m^2 x}{1+m^4x^2} = \frac{0}{1} = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left| \begin{array}{l} \text{približavajmo se} \\ \text{tački } (0,0) \text{ duž} \\ \text{parabole } x=y^2 \end{array} \right| = \lim_{(y^2,y)\to(0,0)} f(y^2,y) = \lim_{(y^2,y)\to(0,0)} \frac{y^4}{y^4+y^4} =$$

$$= \lim_{y\to 0} \frac{y^4}{2y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

Kako dva različita puta vode do dvije različite vrijednosti, dati limes ne postoji.

Tvrdnja 2

Ako dobijemo dvije različite vrijednosti za $\lim\limits_{(x,y)\to(a,b)} f(x,y)$, ako se približavanje tački (a,b) vrši preko različitih nizova tačaka ($(x_n,y_n)\to(a,b)$ kad $n\to\infty$; $(x_n',y_n')\to(a,b)$ kad $n\to\infty$) tada $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ ne postoji.

Pokazati da $\lim\limits_{(x,y)\to(0,0)} \frac{x-y+x^2+y^2}{x+y}$ ne postoji.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left| \begin{array}{l} \text{posmatrajmo niz tački} \\ \left(\tfrac{1}{n},\tfrac{1}{n}\right)\to(0,0) \text{ kad} \\ n\to\infty \end{array} \right| = \lim_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n}-\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^2}}{\frac{1}{n}+\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n^2}}{\frac{2}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \left| \begin{array}{l} \text{posmatrajmo niz tački} \\ \left(\tfrac{2}{n},\tfrac{1}{n}\right)\to(0,0) \text{ kad} \\ n\to\infty \end{array} \right| = \lim_{n\to\infty} f\left(\frac{2}{n},\frac{1}{n}\right) =$$

$$= \lim_{n\to\infty} \frac{\frac{2}{n}-\frac{1}{n}+\frac{4}{n^2}+\frac{1}{n^2}}{\frac{2}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{5}{n^2}}{\frac{3}{n}} = \lim_{n\to\infty} \frac{\frac{5+n}{n^2}}{\frac{3}{n}} = \lim_{n\to\infty} \frac{5+n}{3n} = \frac{1}{3}$$

Granična vrijednost zavisi od načina približavanja ka tački $(0,0)$ pa limes ne postoji.

Tvrdnja 3

Ako je $g(x,y) \le f(x,y) \le h(x,y)$ za $\forall (x,y) \in \mathbb{R}^2$ i vrijedi $\lim_{(x,y)\to(a,b)} g(x,y) = \lim_{(x,y)\to(a,b)} h(x,y) = M$ tada je $\lim_{(x,y)\to(a,b)} f(x,y) = M$.

\# Odrediti $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ ako postoji.

Rj.

$x^2 \le x^2+y^2$ zato što je $y^2 \ge 0$ pa je

$$\frac{x^2}{x^2+y^2} \le 1 \quad \Rightarrow \quad \frac{3|y|x^2}{x^2+y^2} \le 3|y|$$

$$0 \le \left| \frac{3x^2y}{x^2+y^2} \right| \le \frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2} \to 0$$

kad $(x,y)\to(0,0)$

Prema tome $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$

Posljedica tvrdnje 3

Pretpostavimo da je $|f(x,y) - L| \le g(x,y)$ za sve (x,y) u unutrašnjosti nekog kruga sa centrom u (x_0,y_0) osim možda u (x_0,y_0). Ako je

$$\lim_{(x,y)\to(x_0,y_0)} g(x,y) = 0$$

tada $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$.

\# Neka je data f-ja $f: \mathbb{R}^2 \to \mathbb{R}$ definisana na sljedeći način

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Odrediti da li sljedeći limesi postoje i izračunati one limese koji postoje:

a) $\lim_{x\to 0} [\lim_{y\to 0} f(x,y)]$; $\lim_{y\to 0} [\lim_{x\to 0} f(x,y)]$;

b) $\lim_{(x,y)\to(0,0)} f(x,y)$.

Rj. a) $\lim_{x\to 0} [\lim_{y\to 0} f(x,y)] = \lim_{x\to 0} [\lim_{y\to 0} \frac{x^2-y^2}{x^2+y^2}] = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1$

$\lim_{y\to 0} [\lim_{x\to 0} f(x,y)] = \lim_{y\to 0} [\lim_{x\to 0} \frac{x^2-y^2}{x^2+y^2}] = \lim_{y\to 0} \frac{-y^2}{y^2} = \lim_{y\to 0} (-1) = -1$

b) Kako je $\lim_{x\to 0} [\lim_{y\to 0} f(x,y)] \ne \lim_{y\to 0} [\lim_{x\to 0} f(x,y)]$ to $\lim_{(x,y)\to(0,0)} f(x,y)$ ne postoji.

Neka je data f-ja $f: \mathbb{R}^2 \to \mathbb{R}$ definisana na sljedeći način

$$f(x,y) = \begin{cases} \dfrac{(xy)^2}{(xy)^2 + (x-y)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Odrediti da li sljedeći limesi postoje i izračunati one limese koji postoje:

a) $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right]$; $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$;

b) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Rj.

a) $$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{(xy)^2}{(xy)^2+(x-y)^2}\right] =$$

$$= \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

$$\lim_{y\to 0}\left[\lim_{x\to 0} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{x\to 0} \frac{(xy)^2}{(xy)^2+(x-y)^2}\right] = \lim_{y\to 0} \frac{0}{y^2} =$$

$$= \lim_{y\to 0} 0 = 0$$

b) Pokazaćemo da limes $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ ne postoji. Posmatrajmo dva niza tački $M_n\left(\frac{1}{n}, \frac{1}{n}\right)$ i $P_n\left(\frac{2}{n}, \frac{1}{n}\right)$, $n=1,2,\ldots$ koje teže $(0,0)$ kad $n\to\infty$.

$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{\frac{1}{n^4}}{\frac{1}{n^4} + \left(\frac{1}{n}-\frac{1}{n}\right)^2} = \lim_{n\to\infty} \frac{\frac{1}{n^4}}{\frac{1}{n^4}} = \lim_{n\to\infty} 1 = 1$$

$$\lim_{n\to\infty} f\left(\frac{2}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{4/n^4}{4/n^4 + \frac{1}{n^2}} \begin{matrix} /\cdot n^4 \\ /\cdot n^4 \end{matrix} = \lim_{n\to\infty} \frac{4}{4+n^2} = 0$$

Prema tome limes zavisi od načina približavanja ka tački $(0,0)$ pa ne postoji.

Neprekidnost

Definicija

Za f-ju f dvije promjenjive kažemo da je neprekidna u tački (a,b) akko

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Kažemo da je f neprekidna na oblasti D ako je neprekidna u svakoj tački $(a,b)\in D$.

Izračunati $\lim\limits_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Rj.

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2\cdot 2^3 + 1^3\cdot 2^2 + 3\cdot 1 + 2\cdot 2 = 8+4+3+4$$
$$= 19$$

U kojim tačkama je f-ja $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ neprekidna?

Rj. Vidimo da f-ja nije definisana (f-ja nema smisla) u tački $(0,0)$ (primijetimo da je $f(0,0) = \frac{0}{0}$).

Prema tome f-ja u tački $(0,0)$ ima prekid.

F-ja je neprekidna na oblasti D gdje je

$$D = \{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(0,0)\}.$$

Ispitati neprekidnost f-je $g(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$ u tački $(0,0)$.

Rj. F-ja $g(x,y)$ će biti neprekidna u tački $(0,0)$ ako i samo ako je

$$\lim_{(x,y)\to(0,0)} g(x,y)=g(0,0)$$

tj. akko $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}=0.$

U jednom od prethodnih zadataka smo pokazali da $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2-y^2}{x^2+y^2}$ ne postoji.

Prema tome f-ja ima prekid u tački $(0,0)$.

Ispitati neprekidnost f-je $f(x,y)=\begin{cases}\dfrac{3x^2y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$

Rj. F-ja $f(x,y)$ će biti neprekidna u tački $(0,0)$ ako i samo ako je

$$\lim_{(x,y)\to(0,0)}(x,y)=(0,0)$$

tj. akko $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3x^2y}{x^2+y^2}=0.$

jednom od prethodnih zadataka smo pokazali da $\displaystyle\lim_{(x,y)\to(0,0)}\frac{3x^2y}{x^2+y^2}=0$.

Prema tome f-ja $f(x,y)$ je neprekidna u tački $(0,0)$.

Zadaci za vježbu

§ 2. Početno proučavanje funkcije

Oblast definisanosti

2975. Oblast koja leži unutar paralelograma, obrazovanog pravama: $y=0$, $y=2$, $y=\frac{1}{2}x$, $y=\frac{1}{2}x-1$ prikazati pomoću nejednakosti.

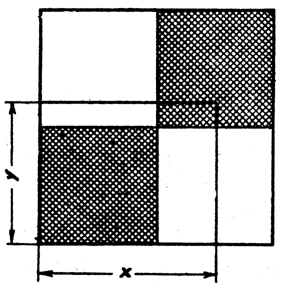
2976. Oblast ograničenu parabolama $y=x^2$ i $x=y^2$ (uključujući granice) definisati nejednakostima.

2977. Opisati pomoću nejednakosti otvorenu oblast, ograničenu jednakostraničnim trouglom stranice a, sa jednim temenom u koordinatnom početku, drugim — na pozitivnom delu x-ose, i trećim — u prvom kvadrantu.

2978. Oblast je ograničena beskonačnim kružnim cilindrom poluprečnika R (isključujući granice), čija je osa paralelna z-osi i prolazi kroz tačku (a, b, c); opisati ovu oblast pomoću nejednakosti.

2979. Oblast ograničenu sferom poluprečnika R sa centrom u tački (a, b, c) (uključujući granicu) definisati pomoću nejednakosti.

2980. Temena pravouglog trougla leže unutar kruga poluprečnika R. Površina S trougla je funkcija njegovih kateta x i y: $S=\varphi(x, y)$: naći: a) oblast definisanosti funkcije φ; b) oblast definisanosti odgovarajućeg analitičkog izraza.



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2981. U loptu poluprečnika R upisana je prava piramida sa pravougaonikom u osnovi. Zapremina V piramide je funkcija osnovnih ivica x i y. Hoće li ova funkcija biti jednoznačno definisana? Sastaviti njoj odgovarajući analitički izraz. i naći oblast definisanosti funkcije i pomenutog analitičkog izraza.

2982. Kvadratna daska se sastoji iz četiri kvadratna polja, dva crna i dva bela kao što je to prikazano na sl. 57; stranica svakog od njih ima dužinu 1. Uočimo pravougaonik čije su stranice x i y paralelne stranicama daske i čiji se jedan ugao poklapa sa njenim crnim uglom. Površina crnog dela ovog pravougaonika biće funkcija od x i y. Naći oblast definisanosti ove funkcije. Izraziti ovu funkciju analitički.

U zadacima 2983—3002 naći oblast definisanosti datih funkcija

2983. $z=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$.

2984. $z=\ln(y^2-4x+8)$.

2985. $z=\frac{1}{R^2-x^2-y^2}$.

2986. $z=\sqrt{x+y}+\sqrt{x-y}$.

2987. $z=\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{x-y}}$.

2988. $z=\arcsin\frac{y-1}{x}$.

2989. $z=\ln xy$.

2990. $z=\sqrt{x-\sqrt{y}}$.

2991. $z=\arcsin\frac{x^2+y^2}{4}+\operatorname{arcsec}(x^2+y^2)$.

2992. $z=\frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$.

2993. $z=\sqrt{\frac{x^2+2x+y^2}{x^2-2x+y^2}}$.

2994. $z=xy+\sqrt{\ln\frac{R^2}{x^2+y^2}}+\sqrt{x^2+y^2-R^2}$.

2995. $z=\operatorname{ctg}\pi(x+y)$.

2996. $z=\sqrt{\sin\pi(x^2+y^2)}$.

2997. $z=\sqrt{x\sin y}$.

2998. $z=\ln x-\ln\sin y$.

2999. $z=\ln[x\ln(y-x)]$.

3000. $z=\arcsin[2y(1+x^2)-1]$.

3001. $u=\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}+\frac{1}{\sqrt{z}}$.

3002. $u=\sqrt{R^2-x^2-y^2-z^2}+\frac{1}{\sqrt{x^2+y^2+z^2-r^2}}$ $(R>r)$.

Granična vrednost. Neprekidnost funkcije

U zadacima 3003—3008 izračunati granične vrednosti datih funkcija uzimajući da nezavisno promenljive na proizvoljan način teže svojim graničnim vrednostima.

3003. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1}$.

3004. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$.

3005. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{\sin(x^3+y^3)}{x^2+y^2}$.

3006. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{1-\cos(x^2+y^2)}{(x^2+y^2)x^2y^2}$.

3007. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{e^{\frac{1}{x^2+y^2}}}{x^4+y^4}$.

3008. $\lim\limits_{\substack{x\to 0\\ y\to 0}}(1+x^2y^2)^{\frac{1}{x^2+y^2}}$.

3009. Uveriti se da funkcija $u=\frac{x+y}{x-y}$ kad $x\to 0$, $y\to 0$ može težiti svakoj graničnoj vrednosti (u zavisnosti od toga kako teže nuli x i y). Navesti primere takvih načina menjanja promenljivih x i y za koje je: 1) $\lim u=1$; 2) $\lim u=2$.

3010. Naći tačke prekida funkcije $z=\frac{2}{x^2+y^2}$. Kako se ponaša funkcija u okolini prekidnih tačaka?

3011. Naći prekidne tačke funkcije $z=\frac{1}{\sin^2\pi x+\sin^2\pi y}$.

3012. U kojim će tačkama funkcija $z=\frac{1}{x-y}$ biti prekidna?

3013. U kojim će tačkama funkcija $z=\frac{1}{\sin\pi x}+\frac{1}{\sin\pi y}$ biti prekidna?

3014. U kojim će tačkama funkcija $z=\frac{y^2+2x}{y^2-2x}$ biti prekidna?

3015*. Ispitati kako stoji sa nepokretnošću datih funkcija za $x=0$, $y=0$:

1) $f(x, y)=\frac{x^2y^2}{x^2+y^2}$; $f(0, 0)=0$.

2) $f(x, y)=\frac{xy}{x^2+y^2}$; $f(0, 0)=0$.

3) $f(x, y)=\frac{x^3y^3}{x^2+y^2}$; $f(0, 0)=0$.

4) $f(x, y)=\frac{1}{x^2+y^2}$; $f(0, 0)=0$.

5) $f(x, y)=\frac{x^4-y^4}{x^4+y^4}$; $f(0, 0)=0$.

6) $f(x, y)=\frac{x^2y^2}{x^4+y^4}$; $f(0, 0)=0$.

Rješenja

2975. $0<y<2$; $-1<y-\frac{1}{2}x<0$. **2976.** $x^2\leqslant y\leqslant\sqrt{x}$.

2977. $0<y<x\sqrt{3}$; $y<(a-x)\sqrt{3}$. **2978.** $(x-a)^2+(y-b)^2<R^2$; $-\infty<z<\infty$.

2979. $(x-a)^2+(y-b)^2+(z-c)^2\leqslant R^2$.

2980. a) $x^2+y^2\leqslant 4R^2$; b) $-\infty<x<\infty$; $-\infty<y<\infty$.

2981. $v=\frac{1}{6}xy\left(2R\pm\sqrt{4R^2-x^2-y^2}\right)$; funkcija nije jednoznačna. Oblast definisanosti funkcije je $x^2+y^2\leqslant 4R^2$; $x>0$, $y>0$. Oblast definisanosti analitičkog izraza je $x^2+y^2\leqslant 4R^2$.

2982.

Za $0\leqslant x\leqslant 1$,	$0\leqslant y\leqslant 1$	$S=xy$;
za $0\leqslant x\leqslant 1$,	$1\leqslant y$	$S=x$;
za $1\leqslant x$	$0\leqslant y\leqslant 1$	$S=y$;
za $1\leqslant x\leqslant 2$,	$1\leqslant y\leqslant 2$	$S=xy-x-y+2$;
za $1\leqslant x\leqslant 2$,	$2\leqslant y$	$S=x$;
za $2\leqslant x$,	$1\leqslant y\leqslant 2$	$S=y$;
za $2\leqslant x$,	$2\leqslant y$	$S=2$;

funkcija nije definisana za $x<0$ i $y<0$.

2983. $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1$. **2984.** $y^2>4x-8$.

2985. Sva ravan izuzev tačaka kružne linije $x^2+y^2=R^2$.

2986. Unutrašnjost desnog pravog ugla koji obrazuju simetrale koordinatnih uglova, uključujući i odgovarajuće delove simetrale, tj.

$$x+y\geqslant 0,\quad x-y\geqslant 0.$$

2987. Ista kao i u zad. 2986, samo bez tačaka na granici oblasti.

2988. Unutrašnjost desnog i levog ugla koje obrazuju prave $y=1+x$ i $y=1-x$, uključujući i te prave, ali bez njihove presečne tačke:

$$1-x\leqslant y\leqslant 1+x\quad (x>0),$$
$$1+x\leqslant y\leqslant 1-x\quad (x<0).$$

(za $x=0$ funkcija nije definisana).

2989. Unutrašnjost prvog i trećeg kvadranta.

2990. Zatvorena oblast između pozitivnog dela apscisne ose i parabole $y=x^2$ (isključujući i granicu):

$$x\geqslant 0,\ y\geqslant 0;\ x^2\geqslant y.$$

2991. Prstenasta oblast između krugova $x^2+y^2=1$ i $x^2+y^2=4$, uključujući i samo krugove: $1\leqslant x^2+y^2<4$.

2992. Deo ravni koji leži unutar parabole $y^2=4x$, između parabole i kruga $x^2+y^2=1$, uključujući luk parabole izuzev njegovog temena, i isključujući luk kruga.

2993. Deo ravni koji leži izvan krugova čiji su poluprečnici jednaki jedinici a centri su im u tačkama $(-1, 0)$ i $(1, 0)$; tačke prvog kruga pripadaju oblasti, tačke drugog ne pripadaju.

2994. Samo tačke kružne linije $x^2+y^2=R^2$.

2995. Sva ravan, izuzev pravih $x+y=n$ (n je ma koji ceo broj, pozitivan, negativan ili nula).

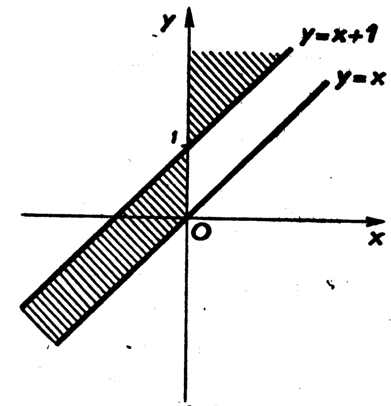
2996. Unutrašnjost kruga $x^2+y^2=1$ i prsten $2n\leqslant x^2+y^2\leqslant 2n+1$ (n je ceo broj), uključujući i granice.

2997. Ako je $x\geqslant 0$, onda je $2n\pi\leqslant y\leqslant(2n+1)\pi$, ako je $x<0$, onda je $(2n+1)\pi\leqslant y\leqslant(2n+2)\pi$, pri čemu je n ceo broj.

2998. $x>0$; $2n\pi<y<2(n+1)\pi$ (n je ceo broj).

2999. Otvorena šrafirana oblast prikazana na sl. 83: za $x>0$ je $y>x+1$; za $x<0$ je $x<y<x+1$.

3000. Deo ravni između krive $y=\frac{1}{1+x^2}$ i njene asimptote, uključujući i granicu.



Sl. 83

3001. $x>0$, $y>0$, $z>0$.

3002. Deo prostora između sfera $x^2+y^2+z^2=r^2$ i $x^2+y^2+z^2=R^2$, uključujući površinu spoljašnje i isključujući površinu unutrašnje sfere.

3003. 2. **3004.** 0. **3005.** 0.

3006. Funkcija nema granične vrednosti kad $x\to 0$, $y\to 0$.

3007. 0. **3008.** 1.

3009. a) $y=0$ ili $y=x^{\alpha}$ ($\alpha>0$), $x\to 0$ na bilo kakav način; b) $y=\frac{x}{3}$, $x\to 0$ na bilo kakav način.

3010. Tačka (0, 0); u blizini ove tačke funkcija može uzimati koliko se god želi velike pozitivne vrednosti.

3011. Sve tačke sa celobrojnim koordinatama.

3012. Na pravoj $y=x$.

3013. Na pravama $x=m$, $y=n$ (m i n su celi brojevi).

3014. Na paraboli $y^2=2x$.

3015. 1) neprekidna; 2) prekidna; neprekidna posebno po x (tj. pri konstantnom y), i posebno po y (tj. pri konstantnom x); 3) neprekidna; 4) prekidna; 5) prekidna; 6) prekidna. Preći na polarne koordinate.

Parcijalni izvodi f-ja više promjenjivih

Posmatrajmo f-ju z dvije promjenjive $z=f(x,y)$.
Parcijalni izvod po x-u označavamo sa z'_x ili sa $\frac{\partial z}{\partial x}$ (delta z po delta x) ili sa f'_x i definišemo

$$z'_x=\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

Parcijalni izvod po y-nu označavamo sa z'_y ili sa $\frac{\partial z}{\partial y}$ (∂ - delta) ili sa f'_y i definišemo

$$z'_y=\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

\# Odrediti parcijalne izvode f-ja

a) $z=x^3+5xy^2-y^3$

b) $u=\frac{x}{y}+\frac{y}{z}-\frac{z}{x}$

c) $v=\sqrt[x]{e^y}$

Rj. a) Kad vadimo izvod po x-u, samo x tumačimo kao promjenjivu, sve ostalo tumačimo kao broj.

$$\frac{\partial z}{\partial x}=3x^2+5y^2.$$

Analogno za y-on $\frac{\partial z}{\partial y}=10xy-3y^2$.

b) $$\frac{\partial u}{\partial x}=\frac{1}{y}-z\cdot\left(\frac{1}{x}\right)'_x=\frac{1}{y}-z\cdot(-1)x^{-2}=\frac{1}{y}+\frac{z}{x^2}$$

$$\frac{\partial u}{\partial y}=x\cdot(-1)y^{-2}+\frac{1}{z}=-\frac{x}{y^2}+\frac{1}{z}$$

$$\frac{\partial u}{\partial z}=y\cdot\left(\frac{1}{z}\right)'_z-\frac{1}{x}=y\cdot(-1)z^{-2}-\frac{1}{x}=-\frac{y}{z^2}-\frac{1}{x}$$

c) $$\frac{\partial v}{\partial x}=\left(e^{\frac{y}{x}}\right)'_x=e^{\frac{y}{x}}\cdot\left(\frac{y}{x}\right)'_x=ye^{\frac{y}{x}}\cdot(x^{-1})'_x=-ye^{\frac{y}{x}}\cdot x^{-2}=-\frac{y}{x^2}e^{\frac{y}{x}}$$

$$\frac{\partial v}{\partial y}=\left(e^{\frac{y}{x}}\right)'_y=e^{\frac{y}{x}}\cdot\left(\frac{y}{x}\right)'_y=\frac{1}{x}e^{\frac{y}{x}}$$

Pronaći vrijednost parcijalnih izvoda datih f-ja u datim tačkama

a) $f(\alpha,\beta)=\cos(m\alpha-n\beta)$, $\alpha=\frac{\pi}{2m}$, $\beta=0$;

b) $z=\ln(x^2-y^2)$, $x=2$, $y=-1$.

Rj. a) $f'_\alpha=-\sin(m\alpha-n\beta)\cdot(m\alpha-n\beta)'_\alpha=-m\sin(m\alpha-n\beta)$

$f'_\beta=-\sin(m\alpha-n\beta)\cdot(m\alpha-n\beta)'_\beta=n\sin(m\alpha-n\beta)$

$f'_\alpha\left(\frac{\pi}{2m},0\right)=-m\sin\frac{\pi}{2}=-m$, $f'_\beta\left(\frac{\pi}{2m},0\right)=n\sin\frac{\pi}{2}=n$

b)

$$z'_x=\frac{1}{x^2-y^2}\cdot 2x$$

$$z'_y=\frac{1}{x^2-y^2}\cdot(-2y)$$

$$z'_x(2,-1)=\frac{1}{4-1}\cdot 2$$

$$z'_y(2,-1)=\frac{1}{4-1}\cdot 2=\frac{2}{3}$$

Naći sve parcijalne izvode prvog reda f-je

a) $z=x^2y^5+3x^3y-7$ c) $z=(2x^2y^2-x+1)^3$ e) $z=\operatorname{arctg}\frac{y}{x}$

b) $z=x^y$ d) $z=\frac{x+y^2}{x^2+y^2+1}$ f) $u=\sqrt{x^2+y^2+z^2}$

g) $u=\ln(x^3-y^2+z^4)$

Rj. a) $z'_x=2xy^5+9x^2y$

$z'_y=x^2\cdot 5y^4+3x^3=5x^2y^4+3x^3$

b) $z'_x=yx^{y-1}$

$z'_y=x^y\ln x$

c) $z'_x=3(2x^2y^2-x+1)^2(4xy^2-1)$

$z'_y=3(2x^2y^2-x+1)^2(4x^2y)=12x^2y(2x^2y^2-x+1)^2$

d)

$$z'_x=\frac{1\cdot(x^2+y^2+1)-(x+y^2)\cdot 2x}{(x^2+y^2+1)^2}=\frac{x^2+y^2+1-2x^2-2xy^2}{(x^2+y^2+1)^2}=\frac{-x^2+y^2+1-2xy^2}{(x^2+y^2+1)^2}$$

$$z'_y=\frac{2y(x^2+y^2+1)-(x+y^2)(2y)}{(x^2+y^2+1)^2}=\frac{2x^2y+\cancel{2y^3}+2y-2xy-\cancel{2y^3}}{(x^2+y^2+1)^2}=\frac{2y(x^2-x+1)}{(x^2+y^2+1)^2}$$

e) $z=\operatorname{arctg}\frac{y}{x}$

$$z'_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{(-1)\cdot y}{\left(1+\frac{y^2}{x^2}\right)\cdot x^2}=\frac{-y}{x^2+y^2}$$

$$z'_y=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{1}{\left(1+\frac{y^2}{x^2}\right)\cdot x}=\frac{x}{x^2+y^2}$$

f) $u=\sqrt{x^2+y^2+z^2}$, $u'_x=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x=\frac{x}{\sqrt{x^2+y^2+z^2}}$

$u'_y=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y=\frac{y}{\sqrt{x^2+y^2+z^2}}$, $u'_z=\frac{z}{\sqrt{x^2+y^2+z^2}}$

g) $u=\ln(x^3-y^2+z^4)$, $u'_x=\frac{3x^2}{x^3-y^2+z^4}$, $u'_y=\frac{-2y}{x^3-y^2+z^4}$, $u'_z=\frac{4z^3}{x^3-y^2+z^4}$

Proveriti da li f-ja $z = x\ln\frac{y}{x}$ zadovoljava jednakost

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

Rj.

$$\frac{\partial z}{\partial x} = 1\cdot\ln\frac{y}{x} + x\cdot\frac{1}{\frac{y}{x}}\cdot\left(\frac{y}{x}\right)'_x = \ln\frac{y}{x} + \frac{x^2}{y}\cdot(-1)y(x)^{-2} = \ln\frac{y}{x} - 1$$

F-ju z možemo napisati i u obliku $z = x(\ln y - \ln x)$

$$\frac{\partial z}{\partial y} = x\cdot\frac{1}{y} = \frac{x}{y}$$

$$x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\ln\frac{y}{x}-1\right) + y\cdot\frac{x}{y} = x\ln\frac{y}{x} - x + x = x\ln\frac{y}{x} = z$$

F-ja $z = x\ln\frac{y}{x}$ zadovoljava datu jednakost.

Ako je $z = x^y\cdot y^x$ dokazati da je

$$x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = z\cdot(x+y+\ln z).$$

Rj.

$$\frac{\partial z}{\partial x} = yx^{y-1}\cdot y^x + x^y\cdot y^x\ln y$$

$$\frac{\partial z}{\partial y} = x^y\ln x\cdot y^x + x^y\cdot x\,y^{x-1}$$

$$x\cdot\frac{\partial z}{\partial x} = xy\,x^{y-1}y^x + x\ln y\,x^y y^x = y\,x^y y^x + x\ln y\,x^y y^x$$

$$y\cdot\frac{\partial z}{\partial y} = y\ln x\cdot x^y y^x + x\cdot x^y y^x$$

$$x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = y\,x^y y^x + \ln y^x\cdot x^y y^x + x^y y^x\ln x^y + x\,x^y y^x =$$

$$= x^y y^x\left(y + \ln(x^y\cdot y^x) + x\right) = z\cdot(x+y+\ln z)$$

što je i trebalo dobiti

Zadaci za vježbu

Naći parcijalne izvode sljedećih f-ja

1. $z = (5x^3y^3+1)^3$
2. $r = \sqrt{ax^2 - by^2}$
3. $v = \ln(x + \sqrt{x^2+y^2})$
4. $\rho = \arcsin\frac{x}{t}$
5. $f(m,n) = (2m)^{3n}$; izračunati f'_m i f'_n u tački $A(\frac{1}{2}; 2)$
6. $\rho(x,y,z) = \sin^2(3x+2y-z)$; izračunati $\rho'_x(1;-1;1)$, $\rho'_y(1;1;4)$, $\rho'_z(-\frac{1}{2};0;-1)$
7. Provjeriti da li f-ja $v = x^y$ zadovoljava jednakost
$$\frac{x}{y}\cdot\frac{\partial v}{\partial x} + \frac{1}{\ln x}\cdot\frac{\partial v}{\partial y} = 2v$$
8. Provjeriti da li f-ja $w = x + \frac{x-y}{y-z}$ zadovoljava jednakost
$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 1.$$

Rješenja:

1. $z'_x = 45x^2y^2(5x^3y^2+1)^2$; $z'_y = 30x^3y(5x^3y^2+1)^2$.

2. $\frac{\partial r}{\partial x} = \frac{ax}{r}$; $\frac{\partial r}{\partial y} = -\frac{by}{r}$.

3. $\frac{\partial v}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}$; $\frac{\partial v}{\partial y} = \frac{y}{(x+\sqrt{x^2+y^2})\sqrt{x^2+y^2}}$.

4. $\frac{\partial \rho}{\partial x} = \frac{|t|}{t\sqrt{t^2-x^2}}$; $\frac{\partial \rho}{\partial t} = -\frac{x}{|t|\sqrt{t^2-x^2}}$.

5. 12; 0.

6. 0; $2\sin 2$; $-\sin(-1)$

Diferenciranje f-ja više promjenjivih

Posmatrajmo f-ju tri promjenjive $u=f(x,y,z)$. Diferencijal f-je u označavamo sa du i računamo po formuli

$$du = d_x u + d_y u + d_z u = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

gdje su d_xu, d_yu, d_zu parcijalni diferencijali f-je u redom po promjenjivim x, y i z.

$$d_xu = \frac{\partial u}{\partial x}dx, \quad d_yu = \frac{\partial u}{\partial y}dy, \quad d_zu = \frac{\partial u}{\partial z}dz.$$

Odrediti totalne diferencijale f-ja

a) $z = 3x^2y^5$ b) $u = 2x^{yz}$ c) $p = \arccos\frac{1}{uv}$

Rj.

a) Parcijalni izvodi su

$$\frac{\partial z}{\partial x} = 6xy^5, \quad \frac{\partial z}{\partial y} = 15x^2y^4$$

Totalni diferencijal je $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ tj.

$$dz = 6xy^5dx + 15x^2y^4dy$$

b) Parcijalni izvodi su

$$\frac{\partial u}{\partial x} = 2yzx^{yz-1}, \quad \frac{\partial u}{\partial y} = 2x^{yz}\ln x\cdot z, \quad \frac{\partial u}{\partial z} = 2yx^{yz}\ln x$$

Totalni diferencijal je

$$du = 2yzx^{yz-1}dx + 2zx^{yz}\ln x\,dy + 2yx^{yz}\ln x\,dz$$

$$= 2x^{yz}\left(\frac{yz}{x}dx + z\ln x\,dy + y\ln x\,dz\right)$$

c) Parcijalni izvodi su

$$\frac{\partial p}{\partial u} = \frac{-1}{\sqrt{1-\left(\frac{1}{uv}\right)^2}}\cdot\left(\frac{1}{uv}\right)'_u = \frac{-1}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}(-1)(uv)^{-2}\cdot v = \frac{|uv|}{u^2v\sqrt{u^2v^2-1}};$$

$$\frac{\partial p}{\partial v} = \frac{-1}{\sqrt{1-\frac{1}{u^2v^2}}}(-1)(uv)^{-2}\cdot u = \frac{u}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}\cdot\frac{1}{u^2v^2} = \frac{|uv|}{uv^2\sqrt{u^2v^2-1}};$$

Totalni diferencijal

$$dp = \frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|uv|}{u^2v}du - \frac{|uv|}{uv^2}dv\right) = \frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|v|\,du}{v\,|u|} - \frac{|u|\,dv}{u\,|v|}\right)$$

Odrediti parcijalne diferencijale f-je $z=\sqrt[3]{x^3+y^3}$.

Rj.

$$z'_x=\frac{\partial z}{\partial x}=\frac{1}{3}(x^3+y^3)^{-\frac{2}{3}}\cdot 3x^2=\frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}$$

$$z'_y=\frac{\partial z}{\partial y}=\frac{1}{3}(x^3+y^3)^{-\frac{2}{3}}\cdot 3y^2=\frac{y^2}{\sqrt[3]{(x^3+y^3)^2}}$$

dobijeni izrazi za parcijalne izvode nisu definisani u tački (0,0). Izvode u toj tački treba odrediti po definiciji

$$z'_x(0,0)=\lim_{\epsilon\to 0}\frac{z(0+\epsilon,0)-z(0,0)}{\epsilon}=\lim_{\epsilon\to 0}\frac{\sqrt[3]{\epsilon^3+0^3}-0}{\epsilon}=\lim_{\epsilon\to 0}1=1$$

$$z'_y(0,0)=\lim_{\epsilon\to 0}\frac{z(0,0+\epsilon)-z(0,0)}{\epsilon}=\lim_{\epsilon\to 0}\frac{\sqrt[3]{0^3+\epsilon^3}-0}{\epsilon}=\lim_{\epsilon\to 0}\frac{\epsilon}{\epsilon}=1$$

f-ja f ima parcijalne izvode u svim tačkama iz oblasti definisanosti. Parcijalni diferencijali su

$$d_x z=\frac{\partial z}{\partial x}dx=\begin{cases}\dfrac{x^2}{\sqrt[3]{(x^3+y^3)^2}}\,dx, & (x,y)\neq(0,0)\\ dx, & (x,y)=(0,0)\end{cases}$$

$$d_y z=\frac{\partial z}{\partial y}dy=\begin{cases}\dfrac{y^2}{\sqrt[3]{(x^3+y^3)^2}}\,dy, & (x,y)\neq(0,0)\\ dy, & (x,y)=(0,0)\end{cases}$$

Odrediti totalni diferencijal f-je $z=\arcsin\frac{x}{y}$ u tački (4,5).

Rj. F-ja je definisana za $\left|\frac{x}{y}\right|<1$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(\frac{x}{y}\right)'_x=\frac{1}{y\sqrt{1-\frac{x^2}{y^2}}}=\frac{1}{\sqrt{y^2-x^2}},\quad \frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)=\frac{-x}{y\sqrt{y^2-x^2}}$$

$$dz=\frac{1}{\sqrt{y^2-x^2}}dx+\frac{-x}{y\sqrt{y^2-x^2}}dy=\frac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$$

Stavljajući u dobijeni izraz $x=4$ i $y=5$ dobijemo $dz=\frac{1}{15}(5dx-4dy)$.

Pomoću totalnog diferencijala približno izračunati $\ln(\sqrt[3]{1,03}+\sqrt[4]{0,98}-1)$.

Rj. Neka je $z=\ln(\sqrt[3]{x}+\sqrt[4]{y}-1)$ gdje je $x=a+\epsilon=1+0,03$ i $y=b+\omega=1-0,02$.
Tada je $z(a,b)=\ln(\sqrt{1}-\sqrt{1}-1)=\ln 1=0$; $z=z(a,b)+\Delta z$.
($\Delta z=f(a+\epsilon,b+\omega)-f(a,b)$ totalni priraštaj f-je u tački (a,b)).
Kako je $\Delta z\approx dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{1}{\sqrt[3]{x}+\sqrt[4]{y}-1}\left(\frac{1}{3\sqrt[3]{x^2}}dx+\frac{1}{4\sqrt[4]{y^3}}dy\right)=$
$=\frac{1}{1}\left(\frac{1}{3}\cdot 0,03-\frac{1}{4}\cdot 0,02\right)=0,005$. Pa $z=z_0+\Delta z\approx 0,0005$.

Naći totalni diferencijal i totalni priraštaj f-je $z=x^2+y^2+xy$ pri prelazu od tačke (1,1) u tačku (1,1; 0,9).

Rj. po definiciji totalnog priraštaja dobijemo

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)=(x+\Delta x)^2+(y+\Delta y)^2+(x+\Delta x)(y+\Delta y)-(x^2+y^2+xy)=$$

$$=x^2+2x\Delta x+\Delta x^2+y^2+2y\Delta y+\Delta y^2+xy+x\Delta y+y\Delta x+\Delta x\Delta y-x^2-y^2-xy$$

$$=2x\Delta x+\Delta x^2+y\Delta x+2y\Delta y+\Delta y^2+x\Delta y+\Delta x\Delta y=(2x+y+\Delta x)\Delta x+(2y+x+\Delta x+\Delta y)\Delta y$$

Ako ubacimo u formulu vrijednosti $x=1$, $y=1$, $\Delta x=1,1-1=0,1$, $\Delta y=0,9-1=-0,1$ dobićemo totalni priraštaj date f-je u tački (1,1)

$$\Delta z=(2+1+0,1)0,1+(2+1+0,1-0,1)\cdot(-0,1)=3,1\cdot 0,1+3\cdot(-0,1)=0,31-0,3=0,01$$

$dz=(2x+y)dx+(2y+x)dy$ $\qquad dz=(2+1)\cdot 0,1+(2+1)\cdot(-0,1)=0,3-0,3=0$

Diferenciranje složenih f-ja

F-ju z nazivamo složenom f-jom od tri nezavisno promjenjive x, y, t ako je ona zadana putem argumenata $u, v, \dots, w$:

$$z = F(u, v, \dots, w)$$

gdje je

$$u = f(x,y,t), \quad v = \varphi(x,y,t), \quad \dots, \quad w = \psi(x,y,t).$$

Slično bi definisali f-ju od n nezavisno promjenjivih.

Parcijalni izvod složene f-je po jednoj od nezavisnih promjenjivih jednak je sumi proizvoda parcijalnog izvoda f-je po njenom argumentu sa parcijalnim izvodom istog argumenta po nezavisnoj promjenjivoj:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} + \dots + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x};$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} + \dots + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y}; \qquad \dots(*)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t} + \dots + \frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial t}.$$

Ako su svi argumenti $u, v, \dots, w$ f-je jedne nezavisno promjenjive x, tada je i z složena f-ja po promjenjivoj x. Izvod takve složene f-je (od jedne nezavisno promjenjive) naziva se <u>totalni izvod</u> i dat je preko formule

$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx} + \dots + \frac{\partial z}{\partial w}\cdot\frac{dw}{dx}. \qquad \dots(**)$$

(dobije se iz formule totalnog diferencijala f-je $z(u,v,w)$ tako što je podjelimo sa dx).

(#) Naći izvode složenih f-ja

a) $y = u^2 e^v$, $u = \sin x$, $v = \cos x$;

b) $p = u^v$, $u = \ln(x-y)$, $v = e^{\frac{x}{y}}$;

c) $z = x \sin v \cos w$, $v = \ln(x^2+1)$, $w = -\sqrt{1-x^2}$.

Rj: a) Primjetimo da je y složena f-ja po nezavisno promjenjivoj x. Koristimo formulu (**)

$$\frac{dy}{dx} = \frac{\partial y}{\partial u}\cdot\frac{du}{dx} + \frac{\partial y}{\partial v}\cdot\frac{dv}{dx} \overset{(\square)}{=} 2ue^v\cos x - u^2 e^v \sin x$$

$$\frac{\partial y}{\partial u} = 2ue^v, \quad \frac{du}{dx} = \cos x, \quad \frac{\partial y}{\partial v} = u^2 e^v, \quad \frac{dv}{dx} = -\sin x \qquad \dots(\square)$$

b) p je složena f-ja dvije promjenjive x, y. Koristimo formulu (*)

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial x} = vu^{v-1}\cdot\frac{1}{x-y} + u^v \ln u \cdot \frac{1}{y} e^{\frac{x}{y}}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial y} = vu^{v-1}\cdot\frac{1}{y-x} + u^v \ln u \left(-\frac{x}{y^2} e^{\frac{x}{y}}\right)$$

c) z je složena f-ja jedne promjenjive x. Koristimo formulu (**).

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx} + \frac{\partial z}{\partial w}\cdot\frac{dw}{dx}$$

$$\frac{dz}{dx} = \sin v \cos w + x \cos v \cos w \cdot \frac{2x}{x^2+1} - x \sin v \sin w \, \frac{x}{\sqrt{1-x^2}}$$

Nađi diferencijal f-je u (nađi du) ako je $u=f(\sqrt{x^2+y^2})$

Rj. $u=f(\sqrt{x^2+y^2})$, uvedimo oznaku $t=\sqrt{x^2+y^2}$.

$$u=f(t)=f(t(x,y)),\qquad du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=f_t'\cdot\frac{\partial t}{\partial x}=f_t'\,\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x f_t'}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y}=f_t'\cdot\frac{\partial t}{\partial y}=f_t'\cdot\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y\cdot f_t'}{\sqrt{x^2+y^2}}$$

$$du=\frac{f'_{\sqrt{x^2+y^2}}(\sqrt{x^2+y^2})(x\,dx+y\,dy)}{\sqrt{x^2+y^2}}$$

Ako je $z=\dfrac{y}{f(x^2-y^2)}$ tada je $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y}=\dfrac{z}{y^2}$.

Dokazati.

Rj.

$$z=\frac{y}{f(\xi)}\quad\text{gdje je}\quad \xi=x^2-y^2$$

$$\frac{\partial z}{\partial x}=\frac{0\cdot f(\xi)-y\cdot\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}}{f^2(\xi)}=\frac{-2xy\,\frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

$$\frac{\partial z}{\partial y}=\frac{1\cdot f(\xi)-y\cdot\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}}{f^2(\xi)}=\frac{f(\xi)+2y^2\frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

Ako je $x^2=v\cdot w$, $y^2=u\cdot w$, $z^2=u\cdot v$ i $f(x,y,z)=F(u,v,w)$

dokazati $x\cdot\dfrac{\partial f}{\partial x}+y\cdot\dfrac{\partial f}{\partial y}+z\cdot\dfrac{\partial f}{\partial z}=u\cdot\dfrac{\partial F}{\partial u}+v\cdot\dfrac{\partial F}{\partial v}+w\cdot\dfrac{\partial F}{\partial w}$.

Rj. $F(u,v,w)=f(x,y,z)=f(\sqrt{v\cdot w},\sqrt{u\cdot w},\sqrt{u\cdot v})$

$$\frac{\partial F}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$=f_x'\cdot 0+f_y'\cdot\frac{\sqrt{w}}{2\sqrt{u}}+f_z'\cdot\frac{\sqrt{v}}{2\sqrt{u}}=f_y'\cdot\frac{\sqrt{w}}{2\sqrt{u}}+f_z'\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$$\frac{\partial F}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{w}}{2\sqrt{v}}+\frac{\partial f}{\partial y}\cdot 0+\frac{\partial f}{\partial z}\cdot\frac{\sqrt{u}}{2\sqrt{v}}=f_x'\cdot\frac{\sqrt{w}}{2\sqrt{v}}+f_z'\cdot\frac{\sqrt{u}}{2\sqrt{v}}$$

$$\frac{\partial F}{\partial w}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{v}}{2\sqrt{w}}+\frac{\partial f}{\partial y}\cdot\frac{\sqrt{u}}{2\sqrt{w}}+\frac{\partial f}{\partial z}\cdot 0=f_x'\cdot\frac{\sqrt{v}}{2\sqrt{w}}+f_y'\cdot\frac{\sqrt{u}}{2\sqrt{w}}$$

$$u\cdot\frac{\partial F}{\partial u}=\frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2}+\frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$v\cdot\frac{\partial F}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2}+\frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$w\cdot\frac{\partial F}{\partial w}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2}+\frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2}$$

Prema tome $x\cdot\dfrac{\partial f}{\partial x}+y\cdot\dfrac{\partial f}{\partial y}+z\dfrac{\partial f}{\partial z}=u\dfrac{\partial F}{\partial u}+v\dfrac{\partial F}{\partial v}+w\cdot\dfrac{\partial F}{\partial w}$

q.e.d.

ISPITNI ZADATAK

Ako je $z=z(x,y)$ i $x+y+z=f(x^2+y^2+z^2)$ provjeriti da li je tačna jednakost

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y.$$

Rj: $z=z(x,y) \Rightarrow z$ je f-ja dvije promjenjive x i y.

$$z=f(x^2+y^2+z^2)-x-y$$

$$t=x^2+y^2+z^2$$

$$s=-x-y$$

$$z=f(t)+s$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}+\frac{\partial s}{\partial x}$$

$$\frac{\partial z}{\partial x}=f'_t\cdot\left(2x+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial x}-f'_t\cdot 2z\frac{\partial z}{\partial x}=f'_t\cdot 2x-1$$

$$\frac{\partial z}{\partial x}=\frac{2xf'_t-1}{1-2zf'_t}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}+\frac{\partial s}{\partial y}$$

$$\frac{\partial z}{\partial y}=f'_t\cdot\left(2y+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial y}-2zf'_t\frac{\partial z}{\partial x}=2yf'_t-1$$

$$\frac{\partial z}{\partial y}=\frac{2yf'_t-1}{1-2zf'_t}$$

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=\frac{(y-z)(2xf'_t-1)}{1-2zf'_t}+\frac{(z-x)(2yf'_t-1)}{1-2zf'_t}=$$

$$=\frac{\cancel{2xyf'_t}-y-2xzf'_t+z+2yzf'_t-z-\cancel{2xyf'_t}+x}{1-2zf'_t}=$$

$$=\frac{(x-y)-2xzf'_t+2yzf'_t}{1-2zf'_t}=\frac{(x-y)+2zf'_t(-x+y)}{1-2zf'_t}=$$

$$=\frac{(x-y)(1-2zf'_t)}{1-2zf'_t}=x-y$$

Ako je $z=\frac{y}{f(x^2-y^2)}$, gdje je f diferencijabilna f-ja, izračunati $\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}$.

Rj: $z=y\,f^{-1}(x^2-y^2)=y\,f^{-1}(u)$, gdje je $u=x^2-y^2$

$$\frac{\partial z}{\partial x}=y(-1)f_u^{-2}(x^2-y^2)\cdot 2x=\frac{-2xy}{f_u^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y}=\left(y\,f^{-1}(u)\right)'_y=1\cdot f^{-1}(u)+y\cdot(-1)f_u^{-2}(u)\cdot(-2y)=$$

$$=\frac{1}{f(x^2-y^2)}+\frac{2y^2}{f_u^2(x^2+y^2)}$$

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{-2y}{f_u^2(x^2+y^2)}+\frac{1}{y\,f(x^2-y^2)}+\frac{2y}{f_u^2(x^2+y^2)}=$$

$$=\frac{1}{y\,f(x^2-y^2)}=\frac{1}{y^2}\cdot\frac{y}{f(x^2-y^2)}=\frac{z}{y^2}$$

prema tome $\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{z}{y^2}$.

(#) Ako je $z = e^y \varphi(y e^{\frac{x^2}{2y^2}})$ gdje je φ diferencijabilna f-ja, dokazati da je $(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz$.

Rj. $z = e^y \varphi(\xi)$, gdje je $\xi(x,y) = y e^{\frac{x^2}{2y^2}}$.

$$\frac{\partial \xi}{\partial x} = y e^{\frac{x^2}{2y^2}} \cdot 2 \cdot \frac{x}{2y^2} = \frac{x}{y} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial y} = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2 y^{-2}\right)'_y = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2 \cdot (-2) y^{-3}\right)$$

$$= e^{\frac{x^2}{2y^2}} - \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial z}{\partial x} = e^y \cdot \frac{\partial \varphi}{\partial \xi} \cdot \frac{\partial \varphi}{\partial x} = \frac{x}{y} e^y e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial z}{\partial y} = e^y \varphi(\xi) + e^y \cdot \frac{\partial \varphi}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} = e^y \varphi(\xi) + e^y e^{\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi} -$$
$$- e^y \cdot \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = (x^2-y^2) \cdot \frac{x}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi} +$$
$$+ xy\left(e^y \varphi(\xi) + e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi} - \frac{x^2}{y^2} e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi}\right)$$

$$= \underline{\frac{x^3}{y} e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}} \underline{\underline{- yx e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} + xy e^y \varphi(\xi) +$$
$$\underline{\underline{+ xy e^{y+\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}}} \underline{- \frac{x^3}{y} e^{y+\frac{x^2}{2y^2}} \cdot \frac{\partial \varphi}{\partial \xi}} =$$

$$= xy e^y \varphi(\xi) = xy e^y \varphi(y e^{\frac{x^2}{2y^2}}) = xyz$$

Parcijalni izvodi i diferencijali višeg reda f-ja dvije i više promjenjivih

Parcijalnim izvodima drugog reda f-je $z = f(x,y)$ nazivamo parcijalnim izvodima njenih parcijalnih izvoda prvog reda. Za parcijalne izvode drugog reda upotrebljavamo ove oznake

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x,y) \qquad \boxed{\partial \ \text{DELTA}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x,y) \quad \text{itd.}$$

Analogno se definiraju i označavaju izvodi viših redova.

Diferencijalom drugog reda f-je $z = f(x,y)$ nazivamo diferencijal diferencijala prvog reda te f-je za fiksirane priraste nezavisnih varijabli.

$$d^2 z = d(dz)$$

Analogno se određuju diferencijali f-je z višega nego drugog reda, na primjer $d^3 z = d(d^2 z)$ i općenito $d^n z = d(d^{n-1} z)$ $(n = 2, 3, \ldots)$

Ako je $z = f(x,y)$ gdje su x i y nezavisne varijable i f-ja f ima neprekidne parcijalne izvode drugog reda, tada se diferencijal drugog reda f-je z računa po formuli

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2.$$

Općenito, kada postoje neprekidne odgovarajuće derivacije, vrijedi simbolička formula

$$d^n z = \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}\right)^n z,$$

koja se formalno razvije po binomnom zakonu.

Naći parcijalne izvode drugog reda f-je

a) $z=e^{-xy}$ c) $u=x^3y+y^3x+z^3y$ e)ˇ $z=\ln tg\frac{x}{y}$

b) $z=x^3+y^3-xy$ d)ˇ $u=\ln(x+y-z)$ f)ˇ $u=\sin(x^2+y+z^3)$

Rj. a) $z=e^{-xy}$

$$\frac{\partial z}{\partial x}=e^{-xy}\cdot(-y)=-ye^{-xy} \qquad \frac{\partial^2 z}{\partial x^2}=(-y)e^{-xy}\cdot(-y)=y^2e^{-xy}$$

$$\frac{\partial z}{\partial y}=e^{-xy}\cdot(-x)=-xe^{-xy} \qquad \frac{\partial^2 z}{\partial y^2}=(-x)e^{-xy}\cdot(-x)=x^2e^{-xy}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=-e^{-xy}-ye^{-xy}(-x)=$$
$$=e^{-xy}(xy-1)$$

b) $z=x^3+y^3-xy$

$$\frac{\partial z}{\partial x}=3x^2-y \qquad \frac{\partial^2 z}{\partial x^2}=6x, \quad \frac{\partial^2 z}{\partial y^2}=6y, \quad \frac{\partial^2 z}{\partial x\partial y}=-1$$

$$\frac{\partial z}{\partial y}=3y^2-x$$

c) $u=x^3y+y^3x+z^2y$

$$\frac{\partial u}{\partial x}=3x^2y+y^3 \qquad \frac{\partial^2 u}{\partial x^2}=6xy, \quad \frac{\partial^2 u}{\partial y^2}=6xy, \quad \frac{\partial^2 u}{\partial z^2}=6yz$$

$$\frac{\partial u}{\partial y}=x^3+3y^2x+z^3 \qquad \frac{\partial^2 u}{\partial x\partial y}=3x^2+3y^2, \quad \frac{\partial^2 u}{\partial x\partial z}=0$$

$$\frac{\partial u}{\partial z}=3z^2y \qquad \frac{\partial^2 u}{\partial y\partial z}=3z^2$$

d) $u=\ln(x+y-z)$

$$\frac{\partial u}{\partial x}=\frac{1}{x+y-z} \qquad \frac{\partial u}{\partial z}=\frac{-1}{x+y-z} \qquad \frac{\partial^2 u}{\partial y^2}=\frac{-1}{(x+y-z)^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x+y-z} \qquad \frac{\partial^2 u}{\partial x^2}=\frac{-1}{(x+y-z)^2} \qquad \frac{\partial^2 u}{\partial z^2}=\frac{-1}{(x+y-z)^2}$$

završiti sami ...

Provjeriti da li vrijedi

a) $u=\ln(x^2+y^2) \implies \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

b) $u=e^{-\alpha x}\cdot\varphi(x-y) \implies \frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\alpha\cdot\frac{\partial u}{\partial y}=\alpha^2 u$

Rj. a)

$$\frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2} \qquad \frac{\partial^2 u}{\partial x^2}=\frac{2(x^2+y^2)-2x\cdot(2x)}{(x^2+y^2)^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2} \qquad \frac{\partial^2 u}{\partial y^2}=\frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2}=\frac{2x^2-2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}+\frac{2x^2-2y^2}{(x^2+y^2)^2}=\frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}=0$$ što je i trebalo dobiti

b) $u=e^{-\alpha x}\cdot\varphi(x-y)$

$$\frac{\partial u}{\partial x}=e^{-\alpha x}\cdot(-\alpha)\varphi(x-y)+e^{-\alpha x}\cdot\varphi'_x=e^{-\alpha x}\left[-\alpha\varphi(x-y)+\varphi'_x\right]$$

$$\frac{\partial^2 u}{\partial x^2}=e^{-\alpha x}\cdot(-\alpha)\left(-\alpha\varphi(x-y)+\varphi'_x\right)+e^{-\alpha x}\left[-\alpha\varphi'_x+\varphi''_{xx}\right]$$
$$=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-\alpha\varphi'_x-\alpha\varphi'_x+\varphi''_{xx}\right)=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}\right)$$

$$\frac{\partial u}{\partial y}=e^{-\alpha x}\cdot\varphi'_y\cdot(-1)=-e^{\alpha x}\varphi'_y$$

$$\frac{\partial^2 u}{\partial y^2}=-e^{\alpha x}\varphi''_{yy}\cdot(-1)=e^{\alpha x}\cdot\varphi''_{yy}$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\alpha\frac{\partial u}{\partial y}=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}-\varphi''_{yy}+2\alpha\varphi'_y\right)=$$ (u slučaju da je $\varphi'_x=\varphi'_y$ i $\varphi''_{xx}=\varphi''_{yy}$) $=\alpha^2e^{-\alpha x}\varphi(x-y)=\alpha^2u$

Nađi parcijalne izvode prvog i drugog reda f-je $z=\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}(x^2+y^2)'_x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}(x^2+y^2)'_y = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{2x}{x^2+y^2}\right)'_x = 2\left(\frac{x}{x^2+y^2}\right)'_x = 2\,\frac{1\cdot(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}$$
$$= 2\cdot\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \left(\frac{2x}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2)-x\cdot 2y}{(x^2+y^2)^2} = 2\cdot\frac{x^2-2xy+y^2}{(x^2+y^2)^2}$$
$$= 2\cdot\frac{(x-y)^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{2y}{x^2+y^2}\right)'_y = 2\left(\frac{y}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}$$
$$= 2\,\frac{x^2-y^2}{(x^2+y^2)^2}$$

Parcijalni izvodi višeg reda složenih f-ja

Ako je $u=\varphi(\xi,\eta)$ pri čemu je $\xi=x+y$, $\eta=x-y$ izračunati izvode $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x\partial y}$, $\frac{\partial^2 u}{\partial y^2}$.

Rj.

$$\frac{\partial u}{\partial x} = \frac{\partial\varphi}{\partial\xi}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial\varphi}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial\varphi}{\partial\xi} + \frac{\partial\varphi}{\partial\eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial\xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial\eta}\right) = \frac{\partial^2\varphi}{\partial\xi^2}\,\frac{\partial\xi}{\partial x} + \frac{\partial^2\varphi}{\partial\xi\partial\eta}\,\frac{\partial\eta}{\partial x} +$$
$$+\frac{\partial^2\varphi}{\partial\eta\partial\xi}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial^2\varphi}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial^2\varphi}{\partial\xi^2} + 2\,\frac{\partial^2\varphi}{\partial\xi\partial\eta} + \frac{\partial^2\varphi}{\partial\eta^2}$$
$$= \left(\frac{\partial\varphi}{\partial\xi} + \frac{\partial\varphi}{\partial\eta}\right)^2$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial\xi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial\eta}\right) = \frac{\partial^2\varphi}{\partial\xi^2}\,\frac{\partial\xi}{\partial y} + \frac{\partial^2\varphi}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial y} +$$
$$+\frac{\partial^2\varphi}{\partial\eta\partial\xi}\cdot\frac{\partial\xi}{\partial y} + \frac{\partial^2\varphi}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial y} = \frac{\partial^2\varphi}{\partial\xi^2} - \frac{\partial^2\varphi}{\partial\eta^2}$$

(#) Ako je $u=\frac{\varphi(x-y)+\psi(x+y)}{x}$, gdje su φ i ψ diferencijabilne f-je izračunati $\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)-x^2\frac{\partial^2 u}{\partial y^2}$.

Rj.

$$u=\frac{1}{x}\left(\varphi(x-y)+\psi(x+y)\right)=x^{-1}\left(\varphi(x-y)+\psi(x+y)\right)$$

$$u'_x=\frac{\partial u}{\partial x}=(-1)x^{-2}\left(\varphi(\underbrace{x-y}_{s})+\psi(\underbrace{x+y}_{t})\right)+\frac{1}{x}\left(\varphi'_s\cdot s'_x+\psi'_t\cdot t'_x\right)=$$

$$=\frac{-1}{x^2}\left[\varphi(x-y)+\psi(x+y)\right]+\frac{1}{x}\left(\varphi'_s\cdot 1+\psi'_t\cdot 1\right)$$

$$x^2\frac{\partial u}{\partial x}=-\varphi(x-y)-\psi(x+y)+x\left(\varphi'_s+\psi'_t\right)$$ gdje su $s=x-y$ i $t=x+y$

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)=-\varphi'_s\cdot 1-\psi'_t\cdot 1+1\cdot(\varphi'_s+\psi'_t)+x\left(\varphi''_{ss}\cdot 1+\psi''_{tt}\cdot 1\right)$$

$$=x\left(\varphi''_{ss}+\psi''_{tt}\right) \quad \dots(1)$$

$$\frac{\partial u}{\partial y}=\frac{1}{x}\left(\varphi'_s\cdot s'_y+\psi'_t\cdot t'_y\right)=\frac{1}{x}\left(\varphi'_s\cdot(-1)+\psi'_t\cdot 1\right)=\frac{1}{x}\left(-\varphi'_s+\psi'_t\right)$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{1}{x}\left(-\varphi''_{ss}\cdot s'_y+\psi''_{tt}\cdot t'_y\right)=\frac{1}{x}\left(\varphi''_{ss}+\psi''_{tt}\right)$$

$$x^2\frac{\partial^2 u}{\partial y^2}=x\left(\varphi''_{ss}+\psi''_{tt}\right) \quad \dots(2)$$

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)-x^2\frac{\partial^2 u}{\partial y^2}\overset{(1)\,i\,(2)}{=}0$$ traženo rešenje

Zadaci za vježbu

§ 3. Izvodi i diferencijali funkcija više promenljivih

Parcijalni izvodi

3032. Zapremina gasa v je funkcija njegove temperature i pritiska: $v=f(p, T)$. Kad pritisak gasa ostaje konstantan, srednjim koeficijentom širenja gasa pri promeni njegove temperature od T_1 do T_2 naziva se veličina $\frac{v_2-v_1}{v(T_2-T_1)}$. Šta treba zvati koeficijentom širenja gasa pri konstantnom pritisku za datu temperaturu T_0?

3033. Temperatura θ u datoj tački A štapa Ox je funkcija apscise x tačke A i vremena t: $\theta=f(x, t)$. Kakav fizički smisao imaju parcijalni izvodi $\frac{\partial\theta}{\partial t}$ i $\frac{\partial\theta}{\partial x}$?

3034. Površina S pravougaonika čija je osnovica b i visina h izražava se obrascem $S=bh$. Naći $\frac{\partial S}{\partial h}$ i $\frac{\partial S}{\partial x}$ i objasniti geometrijski smisao rezultata.

3035. Date su dve funkcije: $u=\sqrt{a^2-x^2}$ (a je konstanta) i $z=\sqrt{y^2-x^2}$. Naći $\frac{du}{dx}$ i $\frac{\partial z}{\partial x}$ i uporediti rezultate.

U zadacima 3036—3084 naći parcijalne izvode datih funkcija po svakoj od nezavisno promenljivih ($x, y, z, u, v, t, \varphi$ i ψ su promenljive veličine).

3036. $z=x-y$.

3037. $z=x^3y-y^3x$.

3038. $\theta=axe^{-t}+bt$ (a, b su konstante).

3039. $z=\frac{u}{v}+\frac{v}{u}$.

3040. $z=\frac{x^3+y^3}{x^2+y^2}$.

3041. $z=(5x^2y-y^3+7)^3$.

3042. $z=x\sqrt{y}+\frac{y}{\sqrt[3]{x}}$.

3043. $z=\ln(x+\sqrt{x^2+y^2})$.

3044. $z=\operatorname{arctg}\frac{x}{y}$.

3045. $z=\frac{1}{\operatorname{arctg}\frac{y}{x}}$.

3046. $z=x^y$.

3047. $z=\ln(x^2+y^2)$.

3048. $z=\ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$.

3049. $z=\arcsin\frac{\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}}$.

3050. $z=\ln\operatorname{tg}\frac{x}{y}$.

3051. $z=e^{-\frac{x}{y}}$.

3052. $z=\ln(x+\ln y)$.

3053. $u=\operatorname{arctg}\frac{v+w}{v-w}$.

3054. $z=\sin\frac{x}{y}\cos\frac{y}{x}$.

3055. $z=\left(\frac{1}{3}\right)^{\frac{y}{x}}$.

3056. $z=(1+xy)^y$.

3057. $z=xy\ln(x+y)$.

3058. $z=x^{xy}$.

3059. $u=xyz$. **3060.** $u=xy+yz+zx$.

3061. $u=\sqrt{x^2+y^2+z^2}$. **3062.** $u=x^3+yz^2+3yx-x+z$.

3063. $w=xyz+yzv+zvk+vxy$. **3064.** $u=e^{x(x^2+y^2+z^2)}$.

3066. $u=\ln(x+y+z)$ **3065.** $u=\sin(x^2+y^2+z^2)$.

3075. $z=\operatorname{arctg}\sqrt{x^y}$. **3067.** $u=x^{\frac{y}{z}}$.

3068. $u=x^{yz}$. **3069.** $f(x, y)=x+y-\sqrt{x^2+y^2}$ u tački (3, 4).

3070. $z=\ln\left(x+\frac{y}{2x}\right)$ u tački (1, 2). **3071.** $z=(2x+y)^{2x+y}$.

3072. $z=(1+\log_y x)^3$. **3073.** $z=xye^{\sin\pi xy}$.

3074. $z=(x^2+y^2)\frac{1-\sqrt{x^2+y^2}}{1+\sqrt{x^2-y^2}}$. **3076.** $z=2\sqrt{\frac{1-\sqrt{xy}}{1+\sqrt{xy}}}$.

3077. $z=\ln[xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}]$.

3078. $z=\sqrt{1-\left(\frac{x+y}{xy}\right)^2}+\arcsin\frac{x+y}{xy}$.

3079. $z=\operatorname{arctg}\left(\operatorname{arctg}\frac{y}{x}\right)\frac{1}{2}\frac{\operatorname{arctg}\frac{x}{y}-1}{\operatorname{arctg}\frac{x}{y}+1}-\operatorname{arctg}\frac{x}{y}$.

3080. $u=\frac{k}{(x^2+y^2+z^2)^2}$. **3081.** $u=\operatorname{arctg}(x-y)^z$.

3082. $u=(\sin x)^{yz}$. **3083.** $u=\ln\frac{1-\sqrt{x^2+y^2+z^2}}{1+\sqrt{x^2+y^2+z^2}}$.

3084. $w=\frac{1}{2}\operatorname{tg}^2(x^2y^2+z^2v^2-xyzv)+\ln\cos(x^2y^2+z^2v^2-xyzv)$.

3085. $n=\frac{\cos(\varphi-2\psi)}{\operatorname{sos}(\varphi+2\psi)}$. Naći $\left(\frac{\partial u}{\partial\psi}\right)_{\substack{\varphi=\frac{\pi}{4}\\ \psi=\pi}}$

3086. $u=\sqrt{az^3-bt^3}$. Naći $\frac{\partial u}{\partial z}$ i $\frac{\partial u}{\partial t}$ za $z=b,\ t=a$.

3087. $z=\frac{x\cos y-y\cos x}{1+\sin x+\sin y}$. Naći $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ za $x=y=0$.

3088. $u=\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}$. Naći $\left(\frac{\partial u}{\partial z}\right)_{\substack{x=0\\ y=0\\ z=\frac{\pi}{4}}}$

3089. $u=\ln(1+x+y^2+z^3)$. Naći $u'_x+u'_y+u'_z$ za $x=y=z=1$.

3090. $f(x, y)=x^3y-y^3x$. Naći $\left(\frac{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}}\right)_{\substack{x=1\\ y=2}}$

3091. Koliki ugao zaklapa tangenta u tački (2, 4, 5) krive $\begin{cases} z=\frac{x^2+y^2}{4} \\ y=4 \end{cases}$ sa pozitivnim pravcem apscisne ose.

3092. Koliki ugao zaklapa tangenta krive $\begin{cases} z=\sqrt{1+x^2+y^2} \\ x=1 \end{cases}$ u tački (1, 1, $\sqrt{3}$) sa pozitivnim pravcem ordinatne ose.

3093. Pod kojim se uglom seku ravne krive po kojima ravan $y=2$ preseca površine $z=x^2+\frac{y^2}{6}$ i $z=\frac{x^2+y^2}{3}$?

Diferencijali. Približna računanja

U zadacima 3094—3097 naći parcijalne diferencijale datih funkcija po svakoj od nezavisno promenljivih.

3094. $z=xy^3-3x^2y^2+2y^4$. **3095.** $z=\sqrt{x^2+y^2}$.

3096. $z=\frac{xy}{x^2+y^2}$. **3097.** $u=\ln(x^3+2y^3-z^3)$.

3098. $z=\sqrt[3]{x+y^2}$. Naći $d_y z$ za $x=2,\ y=5,\ \Delta y=0{,}01$.

3099. $z=\sqrt{\ln xy}$. Naći $d_x z$ za $x=1,\ y=1,\ 2,\ \Delta x=0{,}016$.

3100. $u=p-\frac{qr}{p}+\sqrt{p+q+r}$. Naći $d_p u$ za $p=1,\ q=3,\ r=5,\ \Delta p=0{,}01$.

U zadacima 3101—3109 naći totalne diferencijale datih funkcija

3101. $z=x^2y^4-x^3y^3+x^4y^3$. **3102.** $z=\frac{1}{2}\ln(x^2+y^2)$.

3103. $z=\frac{x+y}{x-y}$. **3104.** $z=\arcsin\frac{x}{y}$.

3105. $z=\sin(xy)$. **3106.** $z=\operatorname{arctg}\frac{x+y}{1-xy}$.

3107. $z=\frac{x^2+y^2}{x^2-y^2}$. **3108.** $z=\operatorname{arctg}(xy)$. **3109.** $u=x^{yz}$.

§ 4. Diferenciranje funkcija

Posredna funkcija

3124. $u=e^{x-2y}$, pri čemu je $x=\sin t$, $y=t^3$; $\dfrac{du}{dt}=?$

3125. $u=z^2+y^2+zy$, $z=\sin t$, $y=e^t$; $\dfrac{du}{dt}=?$

3126. $z=\arcsin(x-y)$, $x=3t$, $y=4t^3$; $\dfrac{dz}{dt}=?$

3127. $z=x^2y-y^2x$, gde je $x=u\cos v$, $y=u\sin v$; $\dfrac{\partial z}{\partial u}=?$ $\dfrac{\partial z}{\partial v}=?$

3128. $z=x^2\ln y$, $x=\dfrac{u}{v}$, $y=3u-2v$; $\dfrac{\partial z}{\partial u}=?$ $\dfrac{\partial z}{\partial v}=?$

3129. $u=\ln(e^x-e^y)$; $\dfrac{\partial u}{\partial x}=?$ Naći $\dfrac{du}{dx}$, Ako je $y=x^3$.

3130. $z=\operatorname{arctg}(xy)$; naći $\dfrac{dz}{dx}$, ako je $y=e^x$.

3131. $u=\arcsin\dfrac{x}{z}$, gde je $z=\sqrt{x^2+1}$; $\dfrac{du}{dx}=?$

3132. $z=\operatorname{tg}(3t+2x^2-y)$, $x=\dfrac{1}{t}$, $y=\sqrt{t}$; $\dfrac{dz}{dt}=?$

3133. $u=\dfrac{e^{ax}(x-z)}{a^2+1}$, $y=a\sin x$, $z=\cos x$; $\dfrac{du}{dx}=?$

3134. $z=\dfrac{xy\operatorname{arctg}(xy+x+y)}{x+y}$; $dz=?$

3135. $z=(x^2+y^2)\,e^{\frac{x^2+y^2}{xy}}$; $\dfrac{\partial z}{\partial x}=?$ $\dfrac{\partial z}{\partial y}=?$ $dz=?$

3136. $z=f(x^2-y^2,\ e^{xy})$; $\dfrac{\partial z}{\partial x}=?$ $\dfrac{\partial z}{\partial y}=?$

3137. Uveriti se da funkcija $z=\operatorname{arctg}\dfrac{x}{y}$, u kojoj je $x=u+v$, $y=u-v$, zadovoljava relaciju

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{u-v}{v^2+u^2}.$$

3138. Uveriti se da funkcija $z=\varphi(x^2+y^2)$, u kojoj je φ diferencijabilna funkcija, zadovoljava relaciju:

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

3139. $u=\sin x+F(\sin y-\sin x)$; uveriti se da je $\dfrac{\partial u}{\partial y}\cos x+\dfrac{\partial u}{\partial x}\cos y=$ $=\cos x\cos y$, ma kakva bila diferencijabilna funkcija F.

3140. $z=\dfrac{y}{f(x^2-y^2)}$, uveriti se da je $\dfrac{1}{x}\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\dfrac{\partial z}{\partial y}=\dfrac{z}{y^2}$, ma kakva bila diferencijabilna funkcija f.

3141. Pokazati da homogena diferencijabilna funkcija $z=F\left(\dfrac{y}{x}\right)$ nultog stepena homogenosti (vidi zad. 2961) zadovoljava relaciju $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=0$.

3142. Pokazati da homogena funkcija $u=x^kF\left(\dfrac{z}{x};\ \dfrac{y}{x}\right)$, k-tog stepena homogenosti, u kojoj je F diferencijabilna funkcija, zadovoljava relaciju

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=ku.$$

3143. Proveriti tvrđenje formulisano u zadatku 3142 na funkciji

$$u=x^5\sin\frac{z^2+y^2}{x^2}.$$

3144. Neka je funkcija $f(x, y)$ diferencijabilna. Dokazati da, ako se promenljive x i y zamene linearnim homogenim funkcijama promenljivih X i Y, onda je tako dobijena funkcija $F(X, Y)$ vezana sa funkcijom $f(x, y)$ sledećom relacijom:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=X\frac{\partial F}{\partial X}+Y\frac{\partial F}{\partial Y}.$$

§ 5. Izvodi višeg reda

3181. $z=x^3+xy^2-5xy^3+y^5$. Uveriti se da je: $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$.

3182. $z=x^y$. Uveriti se da je $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\partial^2 z}{\partial y\partial x}$.

3183. $z=e^x(\cos y+x\sin y)$. Uveriti se da je

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}.$$

3184. $z=\operatorname{arctg}\dfrac{y}{x}$. Uveriti se da je $\dfrac{\partial^3 z}{\partial y^2\partial x}=\dfrac{\partial^3 z}{\partial x\partial y^2}$.

U zadacima 3185—3192 naći $\dfrac{\partial^2 z}{\partial x^2}$, $\dfrac{\partial^2 z}{\partial x\partial y}$, i $\dfrac{\partial^2 z}{\partial y^2}$ za date frnkcije.

3185. $z=\dfrac{1}{3}\sqrt{(x^2+y^2)^3}$.

3186. $z=\ln(x+\sqrt{x^2+y^2})$.

3187. $z=\operatorname{arctg}\dfrac{x+y}{1-xy}$.

3188. $z=\sin^2(ax+by)$.

3189. $z=e^{xe^y}$.

3190. $z=\dfrac{x-y}{x+y}$.

3191. $z=y^{\ln x}$.

3192. $z=\arcsin(xy)$.

3193. $u=\sqrt{x^2+y^2+z^2-2xz}$; $\dfrac{\partial^2 u}{\partial y\partial z}=?$

3194. $z=e^{xy^2}$; $\dfrac{\partial^3 z}{\partial x^2\partial y}=?$

3195. $s=\ln(x^2+y^2)$; $\dfrac{\partial^3 z}{\partial x\partial y^2}=?$

3196. $z=\sin xy$; $\dfrac{\partial^3 z}{\partial x\partial y^2}=?$

3197. $w=e^{xyz}$; $\dfrac{\partial^3 w}{\partial x\partial y\partial z}=?$

3198. $v=x^my^nz^p$; $\dfrac{\partial^6 v}{\partial x\partial y^3\partial z^2}=?$

3199. $z=\ln(e^x+e^y)$; uveriti se da je $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$ i da je

$$\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2=0.$$

3200. $u=e^x(x\cos y-y\sin y)$. Pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

3201. $u=\ln\frac{1}{\sqrt{x^2+y^2}}$; pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

3202. $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$; pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

3203. $r=\sqrt{x^2+y^2+z^2}$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}=\frac{2}{r},\quad \frac{\partial^2(\ln r)}{\partial x^2}+\frac{\partial^2(\ln r)}{\partial y^2}+\frac{\partial^2(\ln r)}{\partial z^2}=\frac{1}{r^2}.$$

3204. Za koje vrednosti konstante a funkcija $v=x^3+axy^2$ zadovoljava jednačinu

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0?$$

3205. $z=\frac{y}{y^2-a^2x^2}$; pokazati da je $\frac{\partial^2 z}{\partial x^2}=a^2\frac{\partial^2 z}{\partial y^2}$.

3206. $v=\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}$; uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+2\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y\,\partial z}+\frac{\partial^2 v}{\partial z\,\partial x}\right)=0.$$

3207. $z=f(x,y)$, $\xi=x+y$, $\eta=x-y$; uveriti se da je

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=4\frac{\partial^2 z}{\partial\xi\,\partial\eta}.$$

3208. $v=x\ln(x+r)-r$, gde je $r^2=x^2+y^2$. Uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{1}{x+r}.$$

3209. Izvesti obrazac za drugi izvod $\frac{d^2 y}{dx^2}$ funkcije y, definisane implicitno jednačinom $f(x,y)=0$.

3210. $y=\varphi(x-at)+\psi(x+at)$. Pokazati da je

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2},$$

ma kakve bile dvaput diferencijabilne funkcije φ i ψ.

3211. $u=\varphi(x)+\psi(y)+(x-y)\psi'(y)$. Uveriti se da je

$$(x-y)\frac{\partial^2 y}{\partial x\,\partial y}=\frac{\partial u}{\partial y}$$

(φ i ψ su dvaput diferencijabilne funkcije).

3212. $z=y\varphi(x^2-y^2)$. Uveriti se da je

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$

(φ je diferencijabilna funkcija).

3213. $r=x\varphi(x+y)+y\psi(x+y)$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2}-2\frac{\partial^2 r}{\partial x\,\partial y}+\frac{\partial^2 r}{\partial y^2}=0$$

(φ i ψ su dvaput diferencijabilne funkcije).

3214. $u=\frac{1}{y}[\varphi(ax+y)+\psi(ax-y)]$. Pokazaii da je

$$\frac{\partial^2 u}{\partial x^2}=\frac{a^2}{y^2}\cdot\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right).$$

3215. $u=\frac{1}{x}[\varphi(x-y)+\psi(x+y)]$. Pokazati da je

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)=x^2\frac{\partial^2 u}{\partial y^2}.$$

3216. $u=xe^y+ye^x$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 y}{\partial y^3}=x\frac{\partial^3 u}{\partial x\,\partial y^2}+y\frac{\partial^3 u}{\partial x^2\,\partial y}.$$

3217. $u=e^{xyz}$. Pokazati da je

$$\frac{\partial^3 y}{\partial x\,\partial y\,\partial z}=xy\frac{\partial^2 u}{\partial x\,\partial y}+2x\frac{\partial u}{\partial x}+u.$$

3218. $u=\ln\frac{x^2-y^2}{xy}$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial x^2\,\partial y}-\frac{\partial^3 u}{\partial x\,\partial y^2}-\frac{\partial^3 u}{\partial y^3}=2\left(\frac{1}{y^3}-\frac{1}{x^3}\right).$$

U zadacima 3219—3224 naći diferencijale drugog reda za date funkcije.

3219. $z=xy^2-x^2y$. 3220. $z=\ln(x-y)$.

3221. $z=\frac{1}{2(x^2+y^2)}$. 3222. $z=x\sin^2 y$.

3223. $z=e^{xy}$. 3224. $u=xyz$.

3225. $z=\sin(2x+y)$. Naći d^3z u tačkama $(0,\pi)$; $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

3226. $u+\sin(x+y+z)$; $d^2u=?$

3227. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; $d^2z=?$

3228. $z^3-3xyz=a^3$; $d^2z=?$

3229. $3x^2y^2+2z^2xy-2zx^3+4zy^3-4=0$. Naći d^2z u tački $(2, 1, 2)$.

Rješenja

3032. $\frac{1}{v}\frac{\partial v}{\partial T}$ za $T=T_0$.

3033. $\frac{\partial\theta}{\partial t}$ — brzina menjanja temperature u datoj tački; $\frac{\partial\theta}{\partial x}$ — brzina menjanja temperature u odnosu na dužinu (duž štapa), u datom trenutku vremena.

3034. $\frac{\partial S}{\partial h}=b$ — brzina menjanja površine u zavisnosti od visine pravougaonika; $\frac{\partial S}{\partial h}=h$ — brzina menjanja površine u zavisnosti od osnovice pravougaonika.

3036. $\frac{\partial z}{\partial x}=1,\ \frac{\partial z}{\partial y}=-1.$

3037. $\frac{\partial z}{\partial x}=3x^2y-y^3;\quad \frac{\partial z}{\partial y}=x^3-3y^2x.$

3038. $\frac{\partial\theta}{\partial x}=ae^{-t};\quad \frac{\partial\theta}{\partial t}=-axe^{-t}+b.$

3039. $\frac{\partial z}{\partial u}=\frac{1}{v}-\frac{v}{u^2};\quad \frac{\partial z}{\partial v}=-\frac{u}{v^2}+\frac{1}{u}.$

3040. $\frac{\partial z}{\partial x}=\frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2};$

$\frac{\partial z}{\partial y}=\frac{y^4+3x^2y^2-2x^3y}{(x^2+y^2)^2}.$

3041. $\frac{\partial z}{\partial x}=30xy\,(5x^2y-y^3+7)^2;$

$\frac{\partial z}{\partial y}=3\,(5x^2y-y^3+7)^2\,(5x^2-3y^2).$

3042. $\frac{\partial z}{\partial x}=\sqrt{y}-\frac{y}{3\sqrt[3]{x^4}};\quad \frac{\partial z}{\partial y}=\frac{x}{2\sqrt{y}}+\frac{1}{\sqrt[3]{x}}.$

3043. $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}};\quad \frac{\partial z}{\partial y}=\frac{y}{x^2+y^2+x\sqrt{x^2+y^2}}.$

3044. $\frac{\partial z}{\partial x}=\frac{y}{x^2+y^2};\quad \frac{\partial z}{\partial y}=-\frac{x}{x^2+y^2}.$

3045. $\frac{\partial z}{\partial x}=\frac{y}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2};\quad \frac{\partial z}{\partial y}=-\frac{x}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2}.$

3046. $\frac{\partial z}{\partial x}=yx^{y-1};\quad \frac{\partial z}{\partial y}=x^y\ln x.$

3047. $\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2};\quad \frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}.$

3048. $\frac{\partial z}{\partial x}=-\frac{2}{\sqrt{x^2+y^2}};\quad \frac{\partial z}{\partial y}=\frac{2x}{y\sqrt{x^2+y^2}}.$

3049. $\frac{\partial z}{\partial x}=\frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}};\quad \frac{\partial z}{\partial y}=-\frac{x^2\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}}.$

3050. $\frac{\partial z}{\partial x}=\frac{2}{y\sin\frac{2x}{y}};\quad \frac{\partial z}{\partial y}=-\frac{2x}{y^2\sin\frac{2x}{y}}.$

3051. $\frac{\partial z}{\partial x}=-\frac{1}{y}e^{-\frac{x}{y}};\quad \frac{\partial z}{\partial y}=\frac{x}{y^2}e^{-\frac{x}{y}}.$

3052. $\frac{\partial z}{\partial x}=\frac{1}{x+\ln y};\quad \frac{\partial z}{\partial y}=\frac{1}{y\,(x+\ln y)}.$

3053. $\frac{\partial u}{\partial v}=-\frac{w}{v^2+w^2};\quad \frac{\partial u}{\partial w}=\frac{v}{v^2+w^2}.$

3054. $\frac{\partial z}{\partial x}=\frac{1}{y}\cos\frac{x}{y}\cos\frac{y}{x}+\frac{y}{x^2}\sin\frac{x}{y}\sin\frac{y}{x};$

$\frac{\partial z}{\partial y}=-\frac{x}{y^2}\cos\frac{x}{y}\cos\frac{y}{x}-\frac{1}{x}\sin\frac{x}{y}\sin\frac{y}{x}.$

3055. $\frac{\partial z}{\partial x}=\frac{y}{x^2}3^{-\frac{y}{x}}\ln 3;\quad \frac{\partial z}{\partial y}=-\frac{1}{x}^{-\frac{y}{x}}\ln 3.$

3056. $\frac{\partial z}{\partial x}=y^2\,(1+xy)^{y-1};\quad \frac{\partial z}{\partial y}=xy\,(1+xy)^{y-1}+(1+xy)^y\ln(1+xy).$

3057. $\frac{\partial z}{\partial x}=y\ln(x+y)+\frac{xy}{x+y};\quad \frac{\partial z}{\partial y}=x\ln(x+y)+\frac{xy}{x+y}.$

3058. $\frac{\partial z}{\partial x}=x^{x^y}x^{y-1}(y\ln x+1);\quad \frac{\partial z}{\partial y}=x^y x^{x^y}\ln^2 x$

3059. $\frac{\partial u}{\partial x}=yz;\quad \frac{\partial u}{\partial y}=xz;\quad \frac{\partial u}{\partial z}=xy.$

3060. $\frac{\partial u}{\partial x}=y+z;\quad \frac{\partial u}{\partial y}=x+z;\quad \frac{\partial u}{\partial z}=x+y.$

3061. $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}};\quad \frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}};\quad \frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}.$

3062. $\frac{\partial u}{\partial x}+3x^2+3y-1;\quad \frac{\partial u}{\partial y}=z^2+3x;\quad \frac{\partial u}{\partial z}=2yz+1.$

3063. $\frac{\partial w}{\partial x}=yz+vz+vy;\quad \frac{\partial w}{\partial y}=xz+zv+vx;\quad \frac{\partial w}{\partial z}=xy+yv+vx;\quad \frac{\partial w}{\partial v}=yz+xz+xy.$

3064. $\frac{\partial u}{\partial x}=(3x^2+y^2+z^2)\,e^{x(x^2+y^2+z^2)};$

$\frac{\partial u}{\partial y}=2xye^{x(x^2+y^2+z^2)};\quad \frac{\partial u}{\partial z}=2xz\,e^{x(x^2+y^2+z^2)}.$

3065. $\frac{\partial u}{\partial x}=2x\cos(x^2+y^2+z^2);\quad \frac{\partial u}{\partial y}=2y\cos(x^2+y^2+z^2);$

$\frac{\partial u}{\partial z}=2z\cos(x^2+y^2+z^2).$

3066. $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}=\frac{1}{x+y+z}.$

3067. $\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1};\quad \frac{\partial u}{\partial y}=\frac{1}{z}x^{\frac{y}{z}}\ln x;\quad \frac{\partial u}{\partial z}=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x.$

3068. $\frac{\partial u}{\partial x}=y^z x^{y^z-1};\quad \frac{\partial u}{\partial y}=zy^{z-1}x^{y^z}\ln x;\quad \frac{\partial u}{\partial z}=y^z x^{y^z}\ln x\ln y.$

3069. $\frac{2}{5},\ \frac{1}{5}.$

3070. $0,\ \frac{1}{4}.$

3071. $\frac{\partial z}{\partial x}=2\,(2x+y)^{2x+y}\,[1+\ln(2x+y)];$

$\frac{\partial z}{\partial y}=(2x+y)^{2x+y}\,[1+\ln(2x+y)].$

3072. $\dfrac{\partial z}{\partial x}=\dfrac{3}{x\ln y}\left(1+\dfrac{\ln x}{\ln y}\right)^2$; $\dfrac{\partial z}{\partial y}=-\dfrac{3\ln x}{y\ln^2 y}\left(1+\dfrac{\ln x}{\ln y}\right)^2$.

3073. $\dfrac{\partial z}{\partial x}=ye^{\sin\pi xy}(1+\pi xy\cos\pi xy)$;

$\dfrac{\partial z}{\partial y}=xe^{\sin\pi xy}(1+\pi xy\cos\pi xy)$.

3074. $\dfrac{\partial z}{\partial x}=\dfrac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2x$;

$\dfrac{\partial z}{\partial y}=\dfrac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2y$.

3075. $\dfrac{\partial z}{\partial x}=\dfrac{y\sqrt{x^y}}{2x(1+x^y)}$; $\dfrac{\partial z}{\partial y}=\dfrac{\sqrt{x^y}\ln x}{2(1+x^y)}$.

3076. $\dfrac{\partial z}{\partial x}=-\dfrac{y}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$; $\dfrac{\partial z}{\partial y}=-\dfrac{x}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$.

3077. $\dfrac{\partial z}{\partial x}=\dfrac{y^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$; $\dfrac{\partial z}{\partial y}=\dfrac{x^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$.

3078. $\dfrac{\partial z}{\partial x}=-\dfrac{1}{x^2}\sqrt{\dfrac{xy-x-y}{xy+x+y}}$; $\dfrac{\partial z}{\partial y}=-\dfrac{1}{y^2}\sqrt{\dfrac{xy-x-y}{xy+x+y}}$.

3079. $\dfrac{\partial z}{\partial x}=\dfrac{y\left[\left(1+\operatorname{arctg}^2\dfrac{y}{x}\right)^2+2\operatorname{arctg}^3\dfrac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\dfrac{y}{x}\right)\left(1+\operatorname{arctg}\dfrac{y}{x}\right)^2}$;

$\dfrac{\partial z}{\partial y}=-\dfrac{x\left[\left(1+\operatorname{arctg}^2\dfrac{y}{x}\right)^2+2\operatorname{arctg}^3\dfrac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\dfrac{y}{x}\right)\left(1+\operatorname{arctg}\dfrac{y}{x}\right)^2}$.

3080. $\dfrac{\partial u}{\partial x}=-\dfrac{4kx}{(x^2+y^2+z^2)^3}$;

$\dfrac{\partial u}{\partial y}=-\dfrac{4ky}{(x^2+y^2+z^2)^3}$;

$\dfrac{\partial u}{\partial z}=-\dfrac{4kz}{(x^2+y^2+z^2)^3}$.

3081. $\dfrac{\partial u}{\partial x}=\dfrac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\dfrac{\partial u}{\partial y}=-\dfrac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\dfrac{\partial u}{\partial z}=\dfrac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}$.

3082. $\dfrac{\partial u}{\partial x}=yz(\sin x)^{yz-1}\cos x$; $\dfrac{\partial u}{\partial y}=z(\sin x)^{yz}\ln\sin x$;

$\dfrac{\partial u}{\partial z}=y(\sin x)^{yz}\ln\sin x$.

3083. $\dfrac{\dfrac{\partial u}{\partial x}}{x}=\dfrac{\dfrac{\partial u}{\partial y}}{y}=\dfrac{\dfrac{\partial u}{\partial z}}{z}=\dfrac{2}{r(r^2-1)}$, gde $r=\sqrt{x^2+y^2+z^2}$.

3084. $\dfrac{\partial w}{\partial x}=(2xy^2-yzv)\operatorname{tg}^3\alpha$; $\dfrac{\partial w}{\partial y}=(2x^2y-xzv)\operatorname{tg}^3\alpha$; $\dfrac{\partial w}{\partial z}=(2zv^2-xyv)\operatorname{tg}^3\alpha$;

$\dfrac{\partial w}{\partial v}=(2z^2v-xyz)\operatorname{tg}^3\alpha$, gde je $\alpha=x^2y^2+z^2v^2-xyzv$.

3085. 4. 3086. $\left(\dfrac{\partial u}{\partial z}\right)_{\substack{z=b\\t=a}}=\dfrac{3b}{2}\sqrt{\dfrac{ab}{b^2-a^2}}$;

$\left(\dfrac{\partial u}{\partial t}\right)_{\substack{z=b\\t=a}}=-\dfrac{3a}{3}\sqrt{\dfrac{ab}{b^2-a^2}}$;

3087. 1 i —1. 3088. $\dfrac{\sqrt{2}}{2}$. 3089. $\dfrac{3}{2}$. 3090. $\dfrac{13}{22}$. 3091. 45°.

3092. 30°. 3093. $\operatorname{arctg}\dfrac{4}{7}$.

3094. $d_x z=(y^3-6xy^2)\,dx$; $d_y z=(3xy^2-6x^2y+8y^3)\,dy$.

3095. $d_x z=\dfrac{x\,dx}{\sqrt{x^2+y^2}}$; $d_y z=\dfrac{y\,dy}{\sqrt{x^2+y^2}}$.

3096. $d_x z=\dfrac{y(y^2-x^2)\,dx}{(x^2+y^2)^2}$; $d_y z=\dfrac{x(x^2-y^2)\,dy}{(x^2+y^2)^2}$.

3097. $d_x u=\dfrac{3x^2\,dx}{x^3+2y^3-z^3}$; $d_y u=\dfrac{6y^3\,dy}{x^3+2y^3-z^3}$; $d_z u=\dfrac{-3z^2\,dz}{x^3+2y^3-z^3}$.

3098. $\dfrac{1}{270}$. 3099. $\approx 0{,}0187$. 3100. $\dfrac{97}{600}$.

3101. $xy\,[(2y^3-3xy^2+4x^2y)\,dx+(4y^2x-3yx^2+2x^3)\,dy]$.

3102. $\dfrac{x\,dx+y\,dy}{x^2+y^2}$. 3103. $\dfrac{2(x\,dy-y\,dx)}{(x-y)^2}$. 3104. $\dfrac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$

3105. $(x\,dy+y\,dx)\cos(xy)$. 3106. $\dfrac{dx}{1+x^2}+\dfrac{dy}{1+y^2}$.

3107. $\dfrac{4xy(x\,dy-y\,dx)}{(x^2-y^2)^2}$. 3108. $\dfrac{x\,dy+y\,dx}{1+x^2y^2}$.

3109. $x^{zy-1}(yz\,dx+zx\ln x\,dx+xy\ln x\,dz)$,

3124. $e^{\sin t-2t^3}(\cos t-6t^2)$. 3125. $\sin 2t+2e^{2t}+e^t(\sin t+\cos t)$.

3126. $\dfrac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$. 3127. $\dfrac{\partial z}{\partial u}=3u^2\sin v\cos v(\cos v-\sin v)$;

$\dfrac{\partial z}{\partial v}=u^3(\sin v+\cos v)(1-3\sin v\cos v)$.

3128. $\dfrac{\partial z}{\partial u}=2\dfrac{u}{v^2}\ln(3u-2v)+\dfrac{3u^2}{v^2(3u-2v)}$;

$\dfrac{\partial z}{\partial v}=-\dfrac{2u^2}{v^3}\ln(3u-2v)-\dfrac{2u^2}{v^2(3u-2v)}$.

3129. $\dfrac{\partial u}{\partial x}=\dfrac{e^x}{e^x+e^y}$; $\dfrac{du}{dx}=\dfrac{e^x+3e^{x^3}x^2}{e^x+e^{x^3}}$.

3130. $\dfrac{dz}{dx}=\dfrac{e^x(x+1)}{1+x^2e^{2x}}$. 3131. $\dfrac{du}{dx}=\dfrac{1}{1+x^2}$.

3132. $\dfrac{dz}{dt}=\left(3-\dfrac{4}{t^3}-\dfrac{1}{2\sqrt{t}}\right)\sec^2\left(3t+\dfrac{2}{t^2}-\sqrt{t}\right)$.

3133. $\dfrac{du}{dx}=e^{ax}\sin x$. 3134. $dz=\dfrac{y^2dx+x^2dy}{(x+y)^2}\operatorname{arctg}(xy+x+y)+\dfrac{xy[(y+1)dx+(x+1)dy]}{(x+y)[1+(xy+x+y)^2]}$.

3135. $\dfrac{e^{\frac{x^2+y^2}{xy}}}{x^2y^2}[(y^4-x^4+2xy^3)x\,dy+(x^4-y^4+2x^3y)y\,dx]$.

3136. $\left.\begin{aligned}\dfrac{\partial z}{\partial x}&=2x\dfrac{\partial f}{\partial u}+ye^{xy}\dfrac{\partial f}{\partial v}.\\ \dfrac{\partial z}{\partial y}&=-2y\dfrac{\partial f}{\partial u}+xe^{xy}\dfrac{\partial f}{\partial v}\end{aligned}\right\}\begin{aligned}u&=x^2-y^2;\\ v&=e^{xy}.\end{aligned}$

3185. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{2x^2+y^2}{\sqrt{x^2+y^2}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^2+2y^2}{\sqrt{x^2+y^2}}$; $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{xy}{\sqrt{x^2+y^2}}$.

3186. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^3+(x^2-y^2)\sqrt{x^2+y^2}}{(x^2+y^2)^{\frac{3}{2}}(x+\sqrt{(x^2+y^2)^2}}$;

$\dfrac{\partial^2 z}{\partial x\partial y}=-\dfrac{y}{(x^2+y^2)^{\frac{3}{2}}}$.

3187. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{2x}{(1+x^2)^2}$; $\dfrac{\partial^2 z}{\partial y^2}=-\dfrac{2y}{(1+y^2)^2}$; $\dfrac{\partial^2 z}{\partial x\partial y}=0$.

3188. $\dfrac{\partial^2 z}{\partial x^2}=2a^2\cos 2(ax+by)$; $\dfrac{\partial^2 z}{\partial y^2}=2b^2\cos 2(ax+by)$;

$\dfrac{\partial^2 z}{\partial x\partial y}=2ab\cos 2(ax+by)$.

3189. $\dfrac{\partial^2 z}{\partial x^2}=e^{xe^y+2y}$; $\dfrac{\partial^2 z}{\partial y^2}=x(1+xe^y)e^{xe^y+y}$; $\dfrac{\partial^2 z}{\partial x\partial y}=(1+xe^y)e^{xe^y+y}$.

3190. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{4y}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{4x}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{2(x-y)}{(x+y)^3}$.

3191. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\ln y(\ln y+1)}{x^2}e^{\ln x\ln y}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\ln x(\ln x-1)}{y^2}e^{\ln x\ln y}$;

$\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\ln x\ln y+1}{xy}e^{\ln x\ln y}$.

3192. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{xy^3}{\sqrt{(1-x^2y^2)^3}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^3y}{\sqrt{(1-x^2y^2)^3}}$; $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{1}{\sqrt{(1-x^2y^2)^3}}$.

3193. $\dfrac{(x-z)y}{\sqrt{(x^2+y^2+z^2-2xz)^3}}$. 3194. $2y^3(2+xy^2))e^{xy^2}$.

3195. $\dfrac{4x(3y^2-x^2)}{(x^2+y^2)^3}$. 3196. $-x(2\sin xy+xy\cos xy)$.

3197. $(x^2y^2z^2+3xyz+1)e^{xyz}$.

3198. $mn(n-1)(n-2)p(p-1)x^{m-1}y^{n-3}z^{p-2}$. 3204. $a=-3$.

3209. $\dfrac{d^2y}{dx^2}=-\dfrac{\dfrac{\partial^2 f}{\partial x^2}\left(\dfrac{\partial f}{\partial y}\right)^2-2\dfrac{\partial^2 f}{\partial x\partial y}\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}+\dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial y}\right)^3}=\dfrac{1}{\left(\dfrac{\partial f}{\partial y}\right)^3}\begin{vmatrix}0 & \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y}\\ \dfrac{\partial f}{\partial y} & \dfrac{\partial^2 f}{\partial x\partial y} & \dfrac{\partial^2 f}{\partial y^2}\end{vmatrix}$

3219. $-2y\,dx^2+4(y-x)\,dx\,dy+2x\,dy^2$. 3220. $-\dfrac{(dx-dy)^2}{(x-y)^2}$.

3221. $\dfrac{(3x^2-y^2)\,dx^2+8xy\,dx\,dy+(3y^2-x^2)\,dy^2}{(x^2+y^2)^3}$.

3222. $2\sin 2y\,dx\,dy+2x\cos 2y\,dy^2$. 3223. $e^{xy}[(y\,dx+y\,dy)^2+2\,dx\,dy]$.

3224. $2(z\,dx\,dy+y\,dx\,dz+x\,dy\,dz)$.

3225. $-\cos(2x+y)(2dx+dy)^3$; $(2dx+dy)^3$; 0.

3226. $-\sin(x+y+z)(dx+dy+dz)^2$.

3227. $-\dfrac{c^4}{z^3}\left[\left(\dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}\right)\dfrac{dx^2}{a^2}+\dfrac{2xy}{a^2b^2}dx\,dy+\left(\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)\dfrac{dy^2}{b^2}\right]$.

3228. $\dfrac{2z[xy^3\,dx^2+(x^2y^2+2xyz^2-z^4)\,dx\,dy+x^3y\,dy^2]}{(z^2-xy)^3}$.

3229. $-31{,}5\,dx^2+206\,dx\,dy-306\,dy^2$. 3230. $\dfrac{d^2y}{dt^2}+y$.

Tejlorov red za f-ju dvije i više promjenjivih

Prisjetimo se Tejlorovog reda za f-ju $f(x)$ jedne promjenjive u tački c:

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k =$$

$$= f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots$$

Za f-ju dvije promjenjive $z=f(x,y)$ vrijedi sljedeći Tejlorov red u tački (p_1,p_2)

$$f(x,y) \sim f(p_1,p_2) + df(p_1,p_2) + \frac{d^2 f(p_1,p_2)}{2!} + \frac{d^3 f(p_1,p_2)}{3!} + \dots$$

gdje je $dx = x-p_1$, $dy = y-p_2$, $dx^2 = (x-p_1)^2$, $dy^2 = (y-p_2)^2, \dots$

Isti red možemo napisati u drugačijem obliku

$$\boxed{f(x,y) \sim f(p_1,p_2) + \sum_{n=1}^{\infty} \frac{1}{n!}\left((x-p_1)\frac{\partial}{\partial x} + (y-p_2)\frac{\partial}{\partial y}\right)^n f(p_1,p_2)}$$

Npr. $\left((x-p_1)\frac{\partial}{\partial x} + (y-p_2)\frac{\partial}{\partial y}\right)^3 f(p_1,p_2) =$

$$= \frac{\partial^3 f}{\partial x^3}(p_1,p_2)(x-p_1)^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}(p_1,p_2)(x-p_1)^3(y-p_2) +$$

$$+ 3\frac{\partial^3 f}{\partial x \partial y^2}(p_1,p_2)(x-p_1)(y-p_2)^2 + \frac{\partial^3 f}{\partial y^3}(p_1,p_2)(y-p_2)^3$$

Kada umjesto tačke $P(p_1,p_2)$ posmatramo tačku $(0,0)$ Tejlorov red postaje Maklorenov red.

Slične formule vrijede za f-je tri i više promjenjivih.

\# F-ju $f(x,y,z) = x^3+y^3+z^3-3xyz$ razložiti po formuli Tejlora u okolini tačke $(1,1,1)$.

Rj.

$$f(x,y,z) = f(p_1,p_2,p_3) + \sum_{k=1}^{n} \frac{1}{k!}\left((x-p_1)\frac{\partial}{\partial x} + (y-p_2)\frac{\partial}{\partial y} + (z-p_3)\frac{\partial}{\partial z}\right)^k f(p_1,p_2,p_3) + R_n(x,y,z)$$

$\frac{\partial f}{\partial x} = 3x^2-3yz$ $\quad \frac{\partial f}{\partial y} = 3y^2-3xz$ $\quad \frac{\partial f}{\partial z} = 3z^2-3xy$ $\quad \frac{\partial^2 f}{\partial x \partial z} = -3y$

$\frac{\partial^2 f}{\partial x^2} = 6x$ $\quad \frac{\partial^2 f}{\partial y^2} = 6y$ $\quad \frac{\partial^2 f}{\partial z^2} = 6z$ $\quad \frac{\partial^2 f}{\partial x \partial y} = -3z$

$\frac{\partial^3 f}{\partial x^3} = 6$ $\quad \frac{\partial^3 f}{\partial y^3} = 6$ $\quad \frac{\partial^3 f}{\partial z^3} = 6$ $\quad \frac{\partial^2 f}{\partial y \partial z} = -3x$

$\frac{\partial^4 f}{\partial x^4} = 0$ $\quad \frac{\partial^4 f}{\partial y^4} = 0$ $\quad \frac{\partial^4 f}{\partial z^4} = 0$ $\quad \frac{\partial^3 f}{\partial x \partial y \partial z} = -3$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$d^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial y^2}dy^2 + \frac{\partial^2 f}{\partial z^2}dz^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx\,dy + 2\frac{\partial^2 f}{\partial x \partial z}dx\,dz + 2\frac{\partial^2 f}{\partial y \partial z}dy\,dz$$

$$d^3 f = \frac{\partial^3 f}{\partial x^3}dx^3 + \frac{\partial^3 f}{\partial y^3}dy^3 + \frac{\partial^3 f}{\partial z^3}dz^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2 dy + 3\frac{\partial^3 f}{\partial x^2 \partial z}dx^2 dz + 3\frac{\partial^3 f}{\partial y^2 \partial x}dy^2 dx$$
$$+ 3\frac{\partial^3 f}{\partial y^2 \partial z}dy^2 dz + 3\frac{\partial^3 f}{\partial z^2 \partial x}dx\,dz^2 + 3\frac{\partial^3 f}{\partial z^2 \partial y} + 6\frac{\partial^3 f}{\partial x \partial y \partial z}dx\,dy\,dz$$

Kako su svi parcijalni izvodi reda većeg od tri jednaki nuli, to je ostatak $R_n = 0$, za $\forall n \geq 3$. Slijedi da formula Tejlora ima oblik

$$f(x,y,z) = f(1,1,1) + df(1,1,1) + \frac{1}{2!}d^2 f(1,1,1) + \frac{1}{3!}d^3 f(1,1,1)$$

gdje su $dx = x-1$, $dy = y-1$, $dz = z-1$.

Izračunajmo u tački $(1,1,1)$ vrijednost f-je i njenih diferencijala:

$f(1,1,1) = 0$, $\quad df(1,1,1) = 0dx + 0dy + 0dz = 0$

$$d^2 f(1,1,1) = 6(x-1)^2 + 6(y-1)^2 + 6(z-1)^2 + 6(x-1)(y-1) + 6(x-1)(z-1) + 6(y-1)(z-1)$$

$$d^3 f(1,1,1) = 6\left((x-1)^3 + (y-1)^3 + (z-1)^3\right) - 18(x-1)(y-1)(z-1)$$

Dobijamo:

$$f(x,y,z) = 3(x-1)^2 + 3(y-1)^2 + 3(z-1)^2 + 3(x-1)(y-1) + 3(x-1)(z-1) + 3(y-1)(z-1)$$
$$+ (x-1)^3 + (y-1)^3 + (z-1)^3 - 3(x-1)(y-1)(z-1)$$

f-ja $f(x,y,z)$ razložena po formuli Tejlora

Funkciju $f(x,y)=2x^2-xy-y^2-6x-3y+5$ razložiti po formuli Tejlora u okolini tačke $(1,-2)$.

Rj: $$f(x_1,x_2)=f(\rho_1,\rho_2)+\sum_{k=1}^{n}\frac{1}{k!}\left((x_1-\rho_1)\frac{\partial}{\partial x_1}+(x_2-\rho_2)\frac{\partial}{\partial x_2}\right)^k f(\rho_1,\rho_2)+R_n(x_1,x_2)$$

$$\frac{\partial f(x,y)}{\partial x}=4x-y-6 \qquad \frac{\partial^2 f(x,y)}{\partial x\partial y}=-1 \qquad \frac{\partial f(x,y)}{\partial y}=-2y-x-3$$

$$\frac{\partial^2 f(x,y)}{\partial x^2}=4 \qquad \frac{\partial^3 f(x,y)}{\partial^2 x\partial y}=0 \qquad \frac{\partial f(x,y)}{\partial^2 y}=-2$$

$$\frac{\partial^3 f(x,y)}{\partial x^3}=0 \qquad \frac{\partial^3 f(x,y)}{\partial x\partial^2 y}=0 \qquad \frac{\partial f(x,y)}{\partial^3 y}=0$$

Data f-ja ima neprekidne parcijalne izvode proizvoljnog reda. Parcijalni izvodi reda većeg od dva su jednaki nuli pa je $R_n(x_1,x_2)=0$ za $\forall n>2$.

Tejlorova formula ima oblik

$$f(x_1,x_2)=f(1,-2)+\frac{1}{1}\left((x_1-1)\frac{\partial}{\partial x_1}+(x_2+2)\frac{\partial}{\partial x_2}\right)f(1,-2)$$

$$+\frac{1}{2}\left((x_1-1)\frac{\partial}{\partial x_1}+(x_2+2)\frac{\partial}{\partial x_2}\right)^2 f(1,-2)=$$

$$=f(1,-2)+\frac{\partial f(1,-2)}{\partial x_1}(x_1-1)+\frac{\partial f(1,-2)}{\partial x_2}(x_2+2)$$

$$+\frac{1}{2}\left(\frac{\partial^2 f(1,-2)}{\partial x_1^2}(x_1-1)^2+2\frac{\partial^2 f(1,-2)}{\partial x_1\partial x_2}(x_1-1)(x_2+2)+\frac{\partial^2 f(1,-2)}{\partial x_2^2}(x_2+2)^2\right)$$

Izračunajmo vrijednost f-je i vrijednosti parcijalnih izvoda u tački $(1,-2)$:

$$f(1,-2)=2\cdot1-1\cdot(-2)-(-2)^2-6\cdot1-3\cdot(-2)+5=2+2-4-6+6+5=5$$

$$\frac{\partial f(1,-2)}{\partial x}=4\cdot1-(-2)-6=4+2-6=0, \qquad \frac{\partial f(1,-2)}{\partial y}=-2\cdot(-2)-1-3=4-4=0$$

$$\frac{\partial^2 f(1,-2)}{\partial x^2}=4, \qquad \frac{\partial^2 f(1,-2)}{\partial x\partial y}=-1, \qquad \frac{\partial^2 f(1,-2)}{\partial y^2}=-2.$$

$$f(x,y)=5+0\cdot(x-1)+0(y+2)+\frac{1}{2}\cdot4(x-1)^2+\frac{1}{2}\cdot2\cdot(-1)(x-1)(y+2)+\frac{1}{2}\cdot(-2)(y+2)^2$$

$$=5+2(x-1)^2-(x-1)(y+2)-(y+2)^2$$

f-ja $f(x,y)$ razložena po formuli Tejlora

Funkciju $f(x,y,z)=2x^3-x^2y+3yz^2+5xy+4xz-3x+y-11$ razviti po Taylorovoj formuli u okolini tačke $(-1,0,1)$.

Rj: F-ja $u=f(x,y,z)$ razložena po formuli Tejlora u okolini tačke (ρ_1,ρ_2,ρ_3):

$$f(x,y,z)=f(\rho_1,\rho_2,\rho_3)+\sum_{k=1}^{n}\frac{1}{k!}\left((x-\rho_1)\frac{\partial}{\partial x}+(y-\rho_2)\frac{\partial}{\partial y}+(z-\rho_3)\frac{\partial}{\partial z}\right)^k f(\rho_1,\rho_2,\rho_3)+R_n$$

$$=f(\rho_1,\rho_2,\rho_3)+\frac{1}{1!}\left[\frac{\partial f(\rho_1,\rho_2,\rho_3)}{\partial x}(x-\rho_1)+\frac{\partial f(\rho_1,\rho_2,\rho_3)}{\partial y}(x-\rho_2)+\frac{\partial f(\rho_1,\rho_2,\rho_3)}{\partial z}(x-\rho_3)\right]+$$

$$+\frac{1}{2!}\left[\frac{\partial^2 f(\rho_1,\rho_2,\rho_3)}{\partial x^2}(x-\rho_1)^2+\frac{\partial^2 f(\rho_1,\rho_2,\rho_3)}{\partial y^2}(x-\rho_2)^2+\frac{\partial^2 f(\rho_1,\rho_2,\rho_3)}{\partial z^2}(x-\rho_3)^2+2\frac{\partial^2 f(\rho_1,\rho_2,\rho_3)}{\partial x\partial y}(x-\rho_1)(y-\rho_2)\right.$$

$$\left.+2\frac{\partial^2 f(\rho_1,\rho_2,\rho_3)}{\partial x\partial z}(x-\rho_1)(z-\rho_3)+2\frac{\partial^2 f(\rho_1,\rho_2,\rho_3)}{\partial y\partial z}(y-\rho_2)(z-\rho_3)\right]+\frac{1}{3!}\left[\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial x^3}(x-\rho_1)^3+\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial y^3}(y-\rho_2)^3\right.$$

$$+\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial z^3}(z-\rho_3)^3+3\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial x^2\partial y}(x-\rho_1)^2(y-\rho_2)+3\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial x^2\partial z}(x-\rho_1)^2(z-\rho_2)+$$

$$+3\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial y^2\partial z}(y-\rho_2)^2(z-\rho_3)+3\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial y^2\partial x}(x-\rho_1)(y-\rho_2)^2+3\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial z^2\partial x}(x-\rho_1)(z-\rho_3)^2$$

$$\left.+3\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial z^2\partial y}(y-\rho_2)(z-\rho_3)^2+6\frac{\partial^3 f(\rho_1,\rho_2,\rho_3)}{\partial x\partial y\partial z}(x-\rho_1)(y-\rho_2)(z-\rho_3)\right]+\ldots+R_n=$$

$$=f(\rho_1,\rho_2,\rho_3)+df(\rho_1,\rho_2,\rho_3)+\frac{1}{2!}d^2f(\rho_1,\rho_2,\rho_3)+\frac{1}{3!}d^3f(\rho_1,\rho_2,\rho_3)+\ldots+R_n$$

$$\frac{\partial f}{\partial x}=6x^2-2xy+5y+4z-3 \qquad \frac{\partial f}{\partial x}(-1,0,1)=6+4-3=7 \qquad \frac{\partial^3 f}{\partial y^2\partial x}=0$$

$$\frac{\partial f}{\partial y}=-x^2+3z^2+5x+1 \qquad \frac{\partial f}{\partial y}(-1,0,1)=-1+3-5+1=-2 \qquad \frac{\partial^3 f}{\partial z^2\partial x}=0$$

$$\frac{\partial f}{\partial z}=6yz+4x \qquad \frac{\partial f}{\partial z}(-1,0,1)=-4 \qquad \frac{\partial^3 f}{\partial x\partial y\partial z}=0 \qquad \frac{\partial^3 f}{\partial z^2\partial y}=6$$

$$f(-1,0,1)=-2-4+3-11=-14 \qquad \frac{\partial^2 f}{\partial x\partial z}=4$$

$$\frac{\partial^2 f}{\partial x^2}=12x-2y \qquad \frac{\partial^2 f}{\partial x^2}(-1,0,1)=-12 \qquad \frac{\partial^2 f}{\partial y\partial z}=6z \qquad \frac{\partial^2 f}{\partial y\partial z}(1,0,1)=6$$

$$\frac{\partial^2 f}{\partial y^2}=0 \qquad \frac{\partial^2 f}{\partial z^2}=6y \qquad \frac{\partial^2 f}{\partial z^2}(-1,0,1)=0 \qquad \frac{\partial^3 f}{\partial x^3}=12 \qquad \frac{\partial^3 f}{\partial y^3}=0 \qquad \frac{\partial^3 f}{\partial z^3}=0$$

$$\frac{\partial^2 f}{\partial x\partial y}=-2x+5 \qquad \frac{\partial^2 f}{\partial x\partial y}(-1,0,1)=2+5=7 \qquad \frac{\partial^3 f}{\partial x^2\partial y}=-2 \qquad \frac{\partial^3 f}{\partial x^2\partial z}=0 \qquad \frac{\partial^3 f}{\partial y^2\partial z}=0$$

$$f(x,y,z) = -14 + 7(x+1) - 2y - 4(z-1) - 6(x+1)^2 + 7(x+1)y + 4(x+1)(z-1) + 6y(z-1) + 2(x+1)^3 - (x+1)^2 y + 3y(z-1)^2$$

f-ja razložena po formuli Tejlora

(#) Razložiti po formuli Maklorena do četvrtog reda zaključno f-ju $f(x,y)=\sqrt{1-x^2-y^2}$

Rj.

$$f(x,y) = f(0,0) + df(0,0) + \frac{1}{2!}d^2f(0,0) + \frac{1}{3!}d^3f(0,0) + \frac{1}{4!}d^4f(0,0)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$d^2f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx\,dy + \frac{\partial^2 f}{\partial y^2}dy^2$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$d^3f = \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dx\,dy^2 + \frac{\partial^3 f}{\partial y^3}dy^3$$

$$d^4f = \frac{\partial^4 f}{\partial x^4}dx^4 + 4\frac{\partial^4 f}{\partial x^3 \partial y}dx^3dy + 6\frac{\partial^4 f}{\partial x^2 \partial y^2}dx^2dy^2 + 4\frac{\partial^4 f}{\partial x \partial y^3}dx\,dy^3 + \frac{\partial^4 f}{\partial y^4}dy^4$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(1-x^2-y^2)^{-\frac{1}{2}}\cdot(-2x) = -x(1-x^2-y^2)^{-\frac{1}{2}}, \quad \frac{\partial^2 f}{\partial x^2} = -(1-x^2-y^2)^{-\frac{1}{2}} + \frac{1}{2}x(1-x^2-y^2)^{-\frac{3}{2}}(-2x),$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(1-x^2-y^2)^{-\frac{1}{2}}\cdot(-2y) = -y(1-x^2-y^2)^{-\frac{1}{2}}, \quad \frac{\partial^2 f}{\partial y^2} = -(1-x^2-y^2)^{-\frac{1}{2}} + \frac{1}{2}y(1-x^2-y^2)^{-\frac{3}{2}}\cdot(-2y),$$

$$\frac{\partial^2 f}{\partial x^2} = -(1-x^2-y^2)^{-\frac{1}{2}} - x^2(1-x^2-y^2)^{-\frac{3}{2}}, \quad \frac{\partial^2 f}{\partial y^2} = -(1-x^2-y^2)^{-\frac{1}{2}} - y^2(1-x^2-y^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{1}{2}x(1-x^2-y^2)^{-\frac{3}{2}}\cdot(-2y) = xy(1-x^2-y^2)^{-\frac{3}{2}}, \qquad dx = x, \quad dy = y$$

$$df(0,0) = 0, \quad d^2f(0,0) = -(x^2+y^2), \quad f(0,0) = 1$$

$$\frac{\partial^3 f}{\partial x^3} = \frac{1}{2}(1-x^2-y^2)^{-\frac{3}{2}}\cdot(-2x) - 2x(1-x^2-y^2)^{-\frac{3}{2}} + \frac{3}{2}x^2(1-x^2-y^2)^{-\frac{5}{2}}\cdot(-2x)$$

$$= -3x(1-x^2-y^2)^{-\frac{3}{2}} - 3x^3(1-x^2-y^2)^{-\frac{5}{2}}$$

$$\frac{\partial^3 f}{\partial y^3} = \frac{1}{2}(1-x^2-y^2)^{-\frac{3}{2}}\cdot(-2y) - 2y(1-x^2-y^2)^{-\frac{3}{2}} + \frac{3}{2}y^2(1-x^2-y^2)^{-\frac{5}{2}}\cdot(-2y)$$

$$= -3y(1-x^2-y^2)^{-\frac{3}{2}} - 3y^3(1-x^2-y^2)^{-\frac{5}{2}}$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} = \frac{1}{2}(1-x^2-y^2)^{-\frac{3}{2}}\cdot(-2y) + \frac{3}{2}x^2(1-x^2-y^2)^{-\frac{5}{2}}\cdot(-2y) = -y(1-x^2-y^2)^{-\frac{3}{2}} - 3x^2y(1-x^2-y^2)^{-\frac{5}{2}}$$

$$\frac{\partial^3 f}{\partial x \partial y^2} = \frac{1}{2}(1-x^2-y^2)^{-\frac{3}{2}}\cdot(-2x) + \frac{3}{2}y^2(1-x^2-y^2)^{-\frac{5}{2}}\cdot(-2x) = -x(1-x^2-y^2)^{-\frac{3}{2}} - 3xy^2(1-x^2-y^2)^{-\frac{5}{2}}$$

$$d^3f(0,0) = 0 \qquad d^4f(0,0) = -3(x^2+y^2)^2$$

$$f(x,y) = 1 - \frac{1}{2}(x^2+y^2) - \frac{1}{8}(x^2+y^2)^2 - \dots$$

razlaganje f-je po formuli Maklorena

Razviti u Maklorenov red f-ju $f(x,y)=\ln(1+x+y)$.

Rj. $$f(x,y)=f(\rho_1,\rho_2)+\sum_{k=1}^{\infty}\frac{1}{k!}\left((x-\rho_1)\frac{\partial}{\partial x}+(y-\rho_2)\frac{\partial}{\partial y}\right)^k f(\rho_1,\rho_2)$$

$$\frac{\partial f}{\partial x}=\frac{1}{1+x+y} \qquad \frac{\partial f}{\partial y}=\frac{1}{1+x+y} \qquad \frac{\partial^2 f}{\partial x\partial y}=\frac{(-1)}{(1+x+y)^2}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{(-1)}{(1+x+y)^2} \qquad \frac{\partial f}{\partial y^2}=\frac{(-1)}{(1+x+y)^2} \qquad \frac{\partial^3 f}{\partial x^2\partial y}=\frac{2}{(1+x+y)^3}$$

$$\frac{\partial^3 f}{\partial x^3}=\frac{2}{(1+x+y)^3} \qquad \frac{\partial^n f}{\partial y^n}=\frac{(-1)^{n-1}(n-1)!}{(1+x+y)^n} \qquad \frac{\partial^3 f}{\partial x\partial y^2}=\frac{2}{(1+x+y)^3}$$

$$\frac{\partial^4 f}{\partial x^4}=\frac{-6}{(1+x+y)^4} \qquad \frac{\partial^4 f}{\partial x^3\partial y}=\frac{-6}{(1+x+y)^4}$$

$$\vdots$$

$$\frac{\partial^n f}{\partial x^n}=\frac{(-1)^{n-1}(n-1)!}{(1+x+y)^n} \qquad n=s+t \quad \frac{\partial^n f}{\partial x^s\partial y^t}=\frac{(-1)^{n-1}(n-1)!}{(1+x+y)^n}$$

$$\left((x-0)\frac{\partial}{\partial x}+(y-0)\frac{\partial}{\partial y}\right)f(0,0)=\frac{\partial f(0,0)}{\partial x}x+\frac{\partial f(0,0)}{\partial y}y=x+y$$

$$\left((x-0)\frac{\partial}{\partial x}+(y-0)\frac{\partial}{\partial y}\right)^2 f(0,0)=\frac{\partial^2 f(0,0)}{\partial x^2}x^2+2\frac{\partial^2 f(0,0)}{\partial x\partial y}xy+\frac{\partial^2 f(0,0)}{\partial y^2}y^2=(-1)(x+y)^n$$

$$\vdots$$

$$\left((x-0)\frac{\partial}{\partial x}+(y-0)\frac{\partial}{\partial y}\right)^n f(0,0)=(-1)^{n-1}(n-1)!\,(x+y)^n, \qquad f(0,0)=0$$

$$f(x,y)=\sum_{n=1}^{\infty}\frac{1}{n!}\cdot(-1)^{n-1}\cdot(n-1)!\,(x+y)^n=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(x+y)^n=$$

$$=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}\binom{n}{k}x^{n-k}y^k=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{n}\frac{n!}{k!(n-k)!}x^{n-k}y^k$$

$$=\sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{(-1)^{n-1}(n-1)!}{k!(n-k)!}x^{n-k}y^k$$

razvoj f-je $f(x,y)=\ln(1+x+y)$ u Maklorenov red

F-ju $f(x,y)=\operatorname{arctg}\frac{x-y}{1+xy}$ razviti u Tejlorov red do članova 4. reda u okolini tačke $(0,0)$. Prikazati izgled opšteg člana.

Rj. F-ja $z=f(x,y)$ razložena po formuli Tejlora u okolini tačke (ρ_1,ρ_2):

$$f(x,y)=f(\rho_1,\rho_2)+\sum_{k=1}^{\infty}\frac{1}{k!}\left((x-\rho_1)\frac{\partial}{\partial x}+(y-\rho_2)\frac{\partial}{\partial y}\right)^k f(\rho_1,\rho_2)=$$

$$=f(\rho_1,\rho_2)+\frac{1}{1!}\left[\frac{\partial f(\rho_1,\rho_2)}{\partial x}(x-\rho_1)+\frac{\partial f(\rho_1,\rho_2)}{\partial y}(y-\rho_1)\right]+\frac{1}{2!}\left[\frac{\partial^2 f(\rho_1,\rho_2)}{\partial x^2}(x-\rho_1)^2+\right.$$

$$\left.+\frac{\partial^2 f(\rho_1,\rho_2)}{\partial x\partial y}(x-\rho_1)(y-\rho_2)+\frac{\partial^2 f(\rho_1,\rho_2)}{\partial y^2}(y-\rho_2)^2\right]+\frac{1}{3!}\left[\frac{\partial^3 f(\rho_1,\rho_2)}{\partial x^3}(x-\rho_1)^3+3\frac{\partial^3 f(\rho_1,\rho_2)}{\partial x^2\partial y}(x-\rho_1)^2(y-\rho_2)\right.$$

$$\left.+3\frac{\partial^3 f(\rho_1,\rho_2)}{\partial x\partial y^2}(x-\rho_1)(y-\rho_2)^2+\frac{\partial^3 f(\rho_1,\rho_2)}{\partial y^2}(y-\rho_2)^3\right]+\frac{1}{4!}\cdot\left[\frac{\partial^4 f(\rho_1,\rho_2)}{\partial x^4}(x-\rho_1)^4+\ldots\right.$$

$$\frac{\partial f}{\partial x}=\frac{1}{1+\left(\frac{x-y}{1+xy}\right)^2}\cdot\frac{(1+xy)-(x-y)\cdot y}{(1+xy)^2}=\frac{(1+xy)^2}{(1+xy)^2+(x-y)^2}\cdot\frac{1+xy-xy+y^2}{(1+xy)^2}=\frac{1+y^2}{1+2xy+x^2y^2+x^2-2xy+y^2}=$$

$$=\frac{1+y^2}{1+x^2+y^2+x^2y^2}=\frac{1+y^2}{1+x^2+y^2(1+x^2)}=\frac{1+y^2}{(1+x^2)(1+y^2)}=\frac{1}{1+x^2}$$

$$\frac{\partial f}{\partial x}(0,0)=1, \quad \frac{\partial f}{\partial y}=\frac{1}{1+\left(\frac{x-y}{1+xy}\right)^2}\cdot\frac{(-1)(1+xy)-(x-y)x}{(1+xy)^2}=\frac{(1+xy)^2}{(1+xy)^2+(x-y)^2}\cdot\frac{-1-xy-x^2+xy}{(1+xy)^2}$$

$$=\frac{(-1)(1+x^2)}{1+2xy+x^2y^2+x^2-2xy+y^2}=\frac{(-1)(1+x^2)}{(1+x^2)(1+y^2)}=\frac{-1}{1+y^2}, \quad \frac{\partial f}{\partial y}(0,0)=-1$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{-2x}{(1+x^2)^2}, \quad \frac{\partial^2 f}{\partial x^2}(0,0)=0, \quad \frac{\partial^2 f}{\partial x\partial y}=0, \quad \frac{\partial^2 f}{\partial y^2}=\frac{2y}{(1+y^2)^2}, \quad \frac{\partial^2 f}{\partial y^2}(0,0)=0$$

$$\frac{\partial^3 f}{\partial x^3}=-2\,\frac{(1+x^2)^2-x\cdot 2(1+x^2)\cdot 2x}{(1+x^2)^4}=-2\,\frac{1+x^2-4x^2}{(1+x^2)^3}=-2\,\frac{1-3x^2}{(1+x^2)^3}, \quad \frac{\partial^3 f}{\partial x^3}(0,0)=-2$$

$$\frac{\partial^3 f}{\partial x^2\partial y}=0, \quad \frac{\partial^3 f}{\partial x\partial y^2}=0, \quad \frac{\partial^3 f}{\partial y^3}=2\,\frac{(1+y^2)^2-y\cdot 2(1+y^2)\cdot 2y}{(1+y^2)^4}=2\,\frac{1+y^2-4y^2}{(1+y^2)^3}=2\,\frac{1-3y^2}{(1+y^2)^3}$$

$$\frac{\partial^3 f}{\partial y^3}(0,0)=2, \quad \frac{\partial^4 f}{\partial x^4}=(-2)\cdot\frac{-6x(1+x^2)^3-(1-3x^2)\cdot 3(1+x^2)^2\cdot 2x}{(1+x^2)^6}=(-2)\frac{-6x-6x^3-6x+18x^3}{(1+x^2)^4}$$

$$=(-2)\frac{-12x+12x^3}{(1+x^2)^4}=(-2)(-12)\frac{x-x^3}{(1+x^2)^4}=-24\,\frac{x(x^2-1)}{(x^2+1)^4}, \quad \frac{\partial^4 f}{\partial x^4}(0,0)=0, \quad \frac{\partial^4 f}{\partial x^3\partial y}=0$$

$$\frac{\partial^4 f}{\partial x^2\partial y^2}=0,\quad \frac{\partial^4 f}{\partial x\partial y^3}=0,\quad \frac{\partial^4 f}{\partial y^4}=2\,\frac{-6y(1+y^2)^3-(1-3y^2)3\cdot(1+y^2)^2\cdot 2y}{(1+y^2)}=$$

$$=2\,\frac{-6y-6y^3-6y+18y^3}{(1+y^2)^4}=2\,\frac{-12y+12y^3}{(1+y^2)^4}=2\cdot 12\,\frac{-y+y^3}{(1+y^2)^4}=24\,\frac{y(y^2-1)}{(y^2+1)^4}$$

$$\frac{\partial^4 f}{\partial y^4}(0,0)=0,\quad f(0,0)=\operatorname{arctg} 0=0$$

$$f(x,y)=\frac{1}{1!}(x-y)+\frac{1}{2!}(0+0+0)+\frac{1}{3!}\left((-2)x^3+2y^3\right)+\frac{1}{4!}\cdot 0+\dots$$

$$=x-y+\frac{-1}{3}(x^3-y^3)+\dots=x-y+\frac{(-1)^1}{3}(x^3-y^3)+\frac{(-1)^2}{5}(x^5-y^5)$$

$$+\dots+\frac{(-1)^n}{2n+1}\left(x^{2n+1}-y^{2n+1}\right)+\dots$$ F-ja $f(x,y)$ razložena po formuli Tejlora

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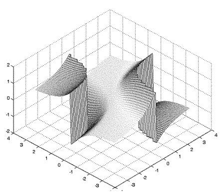
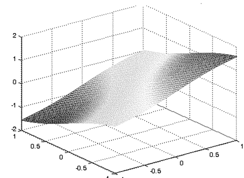
Dodatak.

Grafički prikažimo f-ju $f(x,y)=\operatorname{arctg}\frac{x-y}{1+xy}$ na intervalu



na intervalu [-pi,pi]×[-pi,pi]



na intervalu (-1,1)×(-1,1)

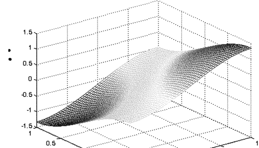
Grafički prikažimo sljedeće polinome:

$$f(x,y)=\sum_{n=0}^{1}\frac{(-1)^n}{2n+1}\left(x^{2n+1}-y^{2n+1}\right)$$

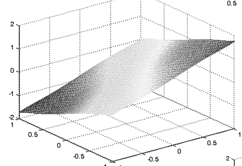
$$f(x,y)=\sum_{n=0}^{2}\frac{(-1)^n}{2n+1}\left(x^{2n+1}-y^{2n+1}\right)$$

$$f(x,y)=\sum_{n=0}^{4}\frac{(-1)^n}{2n+1}\left(x^{2n+1}-y^{2n+1}\right)$$

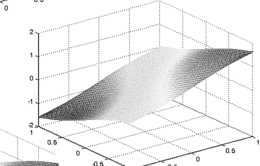
$$F(x,y)=\sum_{n=0}^{10}\frac{(-1)^n}{2n+1}\left(x^{2n+1}-y^{2n+1}\right)$$



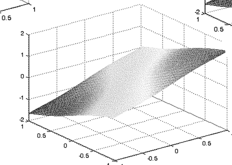
1



2



4



3

Šta možemo primjetiti. Šta bi se desilo da smo uzeli interval $[-\pi,\pi]\times[-\pi,\pi]$ (Kako bi izgledao graf?).

Zadaci za vježbu FI PSI

1. Pokažite da f-ja $z=x\varphi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$ zadovoljava jednačinu $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0$.

2. Pokažite da f-ja $z=e^y\cdot\varphi\left(ye^{\frac{x^2}{2y^2}}\right)$ zadovoljava jednačinu $(x^2-y^2)\frac{\partial z}{\partial x}+xy\cdot\frac{\partial z}{\partial y}=xyz$.

3. Pokažite da f-ja $u=x\varphi(x+y)+y\psi(x+y)$ zadovoljava jednačinu $\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0$

4. F-ju $f(x,y)=x^3+xy^2+xy+x+y$ razložiti po Tejlorovoj formuli u okolini tačke $(1,1)$.

5. Razviti u Tejlorov red f-ju $f(x,y)=e^{x+y}$ do članova 3. reda u okolini tačke $(1,-1)$.

6. Razviti u Maklorenov red f-ju $f(x,y)=\cos x\cos y$ do članova 4. reda.

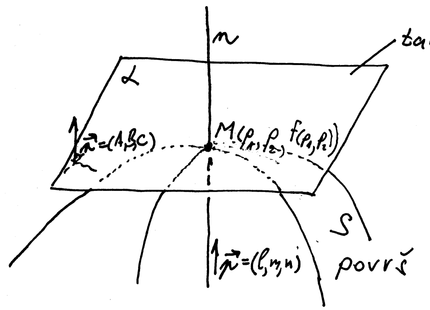
7. Razložiti u Maklorenov red sljedeću f-ju $f(x,y)=e^x\sin y$.

Jednačina tangentne ravni i jednačina normale na površ

Jednačina tangentne ravni (hiperravni) na površ S, čija je jednačina $z=f(x_1,x_2)$, u tački $M(p_1,p_2,f(p_1,p_2))$ (ako je f diferencijabilna u tački (p_1,p_2)) glasi:

$$z-f(p_1,p_2)=f'_{x_1}(p_1,p_2)(x_1-p_1)+f'_{x_2}(p_1,p_2)(x_2-p_2)$$

Može li se uspostaviti sličnost sa jednačinom tangente na krivu liniju $y=f(x)$ u ravni?



$M(p_1,p_2,f(p_1,p_2))$ tačka dodira

n - normala na površ $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

Jednačina normale na površ $z=f(x,y)$ u tački $M(p_1,p_2,f(p_1,p_2))$ (ako je f diferencijabilna u (p_1,p_2)) glasi:

$$\frac{x-p_1}{f'_x(p_1,p_2)}=\frac{y-p_2}{f'_y(p_1,p_2)}=\frac{z-f(p_1,p_2)}{-1}$$

sličnost sa krivom $y=f(x)$ u ravni: $k_1\cdot k_2=-1$, $M(p_1,p_2)$ $y-p_2=f'(p_1)(x-p_1)$

$k_2=\frac{-1}{k_1}$, $y-p_2=\frac{-1}{f'(p_1)}(x-p_1)$

$\frac{x-p_1}{f'(p_1)}=\frac{y-p_2}{-1}$

Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$

$$\alpha:\ F'_x(p_1,p_2,f(p_1,p_2))(x-p_1)+F'_y(p_1,p_2,f(p_1,p_2))(y-p_2)+F'_z(p_1,p_2,f(p_1,p_2))(z-f(p_1,p_2))=0$$

$$n:\ \frac{x-p_1}{F'_x(p_1,p_2,f(p_1,p_2))}=\frac{y-p_2}{F'_y(p_1,p_2,f(p_1,p_2))}=\frac{z-f(p_1,p_2)}{F'_z(p_1,p_2,f(p_1,p_2))}$$

Naći jednačinu tangentne ravni i normale na površ

a) $z=\frac{x^2}{2}-y^2$ u tački $M(2,-1,1)$

b) $3xyz-z^3=a^3$ u tački za koju je $x=0$, $y=a$

c) $z=x^2+2y^2$ u tački $A(1,1,3)$

d) $z=\operatorname{arc\,tg}\frac{y}{x}$ u tački $(1,1,\frac{\pi}{4})$ → rj. $\alpha:\ z=\frac{\pi}{4}-\frac{1}{2}(x-y)$, $n:\ \frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-\frac{\pi}{4}}{2}$

e) $z=\sqrt{169-x^2-y^2}$ u tački $(3,4,z(3,4))$ → rj. $\alpha:\ 3x+4y+12z-169=0$, $n:\ \frac{x-3}{3}=\frac{y-4}{4}=\frac{z-12}{12}$

f) $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{8}=0$ u tački $M(4,3,4)$. → rj. $\alpha:\ 3x+4y-6z=0$, $n:\ \frac{x-4}{3}=\frac{y-3}{4}=\frac{z-4}{-6}$

g) $x^2+y^2+z^2=2Rz$ u tački $(R\cos\alpha, R\sin\alpha, R)$ $(R>0)$.

Rj. a) $z=f(x,y)$, $z-f(p_1,p_2)=f'_x(p_1,p_2)(x-p_1)+f'_y(p_1,p_2)(y-p_2)$ jedn. tang. ravni

$z=\frac{x^2}{2}-y^2$, $z'_x=x$, $z'_x(2,-1)=2$, $\frac{\partial z}{\partial y}=-2y$, $z'_y(2,-1)=2$

$M(2,-1,1)$, $f(2,-1)=1$

$z-1=2(x-2)+2(y+1)$

$2x+2y-z-1$ jednačina tangentne ravni

$$\frac{x-p_1}{f'_x(p_1,p_2)}=\frac{y-p_2}{f'_y(p_1,p_2)}=\frac{z-f(p_1,p_2)}{-1} \Rightarrow \frac{x-2}{2}=\frac{y+1}{2}=\frac{z-1}{-1}$$ jedn. normale

b) Nađimo tačku dodira tangentne ravni i površi

$x=0$, $y=a$, $3xyz-z^3=a^3 \Rightarrow -z^3=a^3 \Rightarrow z=-a$

Tačka dodira je $M(0,a,-a)$

$F'_x=3yz \Rightarrow F'_x(0,a,-a)=-3a^2$

$F'_y=3xz \Rightarrow F'_y(0,a,-a)=0$

$F'_z=3xy-3z^2 \Rightarrow F'_z(0,a,-a)=-3a^2$

$\alpha:\ F'_x(p_1,p_2,f(p_1,p_2))(x-p_1)+F'_y(p_1,p_2,f(p_1,p_2))(y-p_2)+F'_z(p_1,p_2,f(p_1,p_2))(z-f(p_1,p_2))=0$

$-3a^2(x-0)+0(y-a)+(-3a^2)(z-(-a))=0 \Rightarrow -3a^2x-3a^2z-3a^3=0$

tj. $x+z+a=0$ jedn. tang. ravni

$$\frac{x-0}{-3a^2}=\frac{y-a}{0}=\frac{z+a}{-3a^2} \Rightarrow \frac{x}{1}=\frac{y-a}{0}=\frac{z+a}{1}$$ jednačina normale

c) rj. $\alpha:\ 2x+4y-z-3=0$, $n:\ \frac{x-1}{2}=\frac{y-1}{4}=\frac{z-3}{-1}$

g) rj. $\alpha:\ x\cos\alpha+y\sin\alpha-R=0$, $n:\ \frac{x-R\cos\alpha}{\cos\alpha}=\frac{y-R\sin\alpha}{\sin\alpha}=\frac{z-R}{0}$

Na površ $x^2+2y^2+3z^2=21$ postaviti tangentnu ravan paralelnu ravni $x+4y+6z=0$.

Rj. $\beta: Ax+By+Cz+D=0$

$\beta: ?$ $\quad \alpha \parallel \beta$

$\alpha: x+4y+6z=0$

$\vec{n}_\alpha=(1,4,6)$, $\quad \vec{n}_\beta \parallel \vec{n}_\alpha$

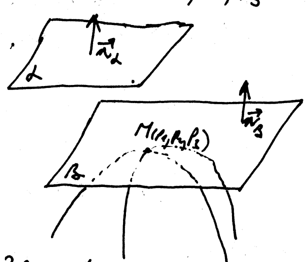
Treba nam tačka dodira tražene tangentne ravni sa površi $x^2+2y^2+3z^2=21$.

$$F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

$F'_x=2x$

$F'_y=4y$

$F'_z=6z$



$$n: \frac{x-p_1}{F'_x(p_1,p_2,p_3)}=\frac{y-p_2}{F'_y(p_1,p_2,p_3)}=\frac{z z_0}{F'_z(p_1,p_2,p_3)}$$

Vektor normale tražene tangentne ravni je

$$\vec{n}_\beta=(2p_1, 4p_2, 6p_3)$$

$$\vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \frac{2p_1}{1}=\frac{4p_2}{4}=\frac{6p_3}{6} \Rightarrow 2p_1=p_2=p_3$$

odredimo p_1, p_2 i p_3:

$p_1^2+2\cdot 4p_1^2+3\cdot 4p_1^2=21$

$21p_1^2=21$

$p_1=\pm 1 \Rightarrow p_2=p_3=\pm 2$

I rješenje: $p_1=-1,\ p_2=p_3=-2$

$-2(x+1)-8(y+2)-12(z+2)=0$

$-2x-8y-12z=42$

$x+4y+6z=-21$

II rješenje $p_1=1,\ p_2=p_3=2$

$2(x-1)+8(y-2)+12(z-2)=0$

$2x+8y+12z-42=0 \quad /:2$

$x+4y+6z=21$

jednačina tražene tangentne ravni

Odrediti jednačine normale i jednačinu tangentne ravni površi $z=\sqrt{169-x^2-y^2}$ u tački $(3,4,z(3,4))$.

Rj. $z(3,4)=\sqrt{169-9-16}=\sqrt{144}=12$

$M(3,4,12)$

jednačina tangentne ravni i normale na površ $z=f(x,y)$ u tački $M(p_1,p_2,p_3)$: $\quad z-p_3=z'_x(p_1,p_2)(x-p_1)+z'_y(p_1,p_2)(y-p_2)$

$$\frac{x-p_1}{z'_x(p_1,p_2)}=\frac{y-p_2}{z'_y(p_1,p_2)}=\frac{z-p_3}{-1}$$

$$=\frac{1}{2\sqrt{169-x^2-y^2}}(-2x)=\frac{-x}{\sqrt{169-x^2-y^2}} \Rightarrow z'_x(3,4)=\frac{-3}{\sqrt{169-25}}=\frac{-3}{12}=-\frac{1}{4}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{169-x^2-y^2}}(-2y)=\frac{-y}{\sqrt{169-x^2-y^2}} \Rightarrow z'_y(3,4)=\frac{-4}{12}=-\frac{1}{3}$$

$z-12=-\frac{1}{4}(x-3)-\frac{1}{3}(y-4) \quad /\cdot 12$

$12z-144=-3(x-3)-4(y-4)$

$3x+4y+12z-144-9-16=0$

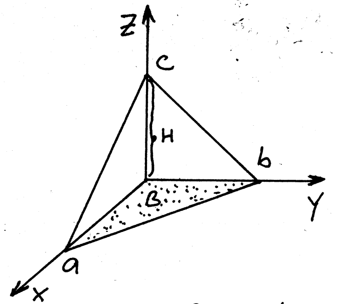
$3x+4y+12z-169=0$ jednačina tangentne ravni na površ z

$$\frac{x-3}{-\frac{1}{4}}=\frac{y-4}{-\frac{1}{3}}=\frac{z-12}{-1} \quad /\cdot\left(\frac{1}{-12}\right)$$

$$\frac{x-3}{3}=\frac{y-4}{4}=\frac{z-12}{12}$$ jednačina normale na površ z

Dokazati da tangentne ravni površi $z=\frac{1}{xy}$ tvore s koordinatnim ravnima piramide konstantne zapremine.

Rj. Jednačina tangentne ravni na površ $z=f(x,y)$ u tački $M(\rho_1,\rho_2,\rho_3)$: $z-\rho_3=z'_x(\rho_1,\rho_2)(x-\rho_1)+z'_y(\rho_1,\rho_2)(y-\rho_2)$



$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ kanonični oblik jednačine ravni gdje su a, b i c odsječci koje ravan odsjeca na koordinatnim osama

$$V_{piramide}=\frac{B\cdot H}{3}=\frac{\frac{a\cdot b}{2}\cdot c}{3}=\frac{a\cdot b\cdot c}{6}$$

$$\frac{\partial z}{\partial x}=\frac{1}{y}\cdot\frac{-1}{x^2}=\frac{-1}{x^2y} \Rightarrow z'_x(\rho_1,\rho_2)=\frac{-1}{\rho_1^2\rho_2}$$

$$\frac{\partial z}{\partial y}=\frac{1}{x}\cdot\frac{-1}{y^2}=\frac{-1}{xy^2} \Rightarrow z'_y(\rho_1,\rho_2)=\frac{-1}{\rho_1\rho_2^2}$$

$$\rho_3=f(\rho_1,\rho_2)=\frac{1}{\rho_1\rho_2}$$

$$z-\frac{1}{\rho_1\rho_2}=\frac{-1}{\rho_1^2\rho_2}(x-\rho_1)+\frac{-1}{\rho_1\rho_2^2}(y-\rho_2) \quad /\cdot\rho_1^2\rho_2^2$$

$$\rho_1^2\rho_2^2 z-\rho_1\rho_2=-\rho_2(x-\rho_1)-\rho_1(y-\rho_2)$$

$$\rho_1^2\rho_2^2 z+\rho_2 x+\rho_2 y=\rho_1\rho_2+\rho_1\rho_2+\rho_1\rho_2 \quad /\cdot\frac{1}{\rho_1\rho_2}$$

$$\frac{x}{\rho_1}+\frac{y}{\rho_1}+\rho_1\rho_2 z=3 \quad /\cdot\frac{1}{3}$$

$$\frac{x}{3\rho_1}+\frac{y}{3\rho_2}+\frac{z}{\frac{3}{\rho_1\rho_2}}=1 \Rightarrow V_{piramide}=\frac{\cancel{3}\rho_1\cdot 3\rho_2\cdot\frac{3}{\rho_1\rho_2}}{\cancel{6}\,2}=\frac{9}{2}$$

zapremina piramide za sve tangentne ravni na površ

Nađite udaljenost ishodišta koordinatnog sistema od tangentne ravni (helikoida) $y=x\,tg\frac{z}{a}$ u tački $(a,a,\frac{\pi a}{4})$.

Rj. $F'_x(\rho_1,\rho_2,\rho_3)(x-\rho_1)+F'_y(\rho_1,\rho_2,\rho_3)(y-\rho_3)+F'_z(\rho_1,\rho_2,\rho_3)(z-\rho_3)=0$

jednačina tangentne ravni na površ $F(x,y,z)=0$

$$y-x\,tg\frac{z}{a}=0$$

$$\frac{\partial F}{\partial x}=-tg\frac{z}{a} \Rightarrow F'_x(a,a,\tfrac{\pi a}{4})=-tg\frac{\frac{\pi a}{4}}{a}=-tg\frac{\pi}{4}=-1$$

$$\frac{\partial F}{\partial y}=1 \Rightarrow F'_y(a,a,\tfrac{\pi a}{4})=1$$

$$\frac{\partial F}{\partial z}=\frac{-x}{\cos^2\frac{z}{a}}\cdot\frac{1}{a}=\frac{-x}{a\cos^2\frac{z}{a}} \Rightarrow F'_z(a,a,\tfrac{\pi a}{4})=\frac{-a}{a\cos^2\frac{\pi}{4}}=\frac{-1}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$F'_z(a,a,\tfrac{\pi a}{4})=-2$$

$$-1(x-a)+1\cdot(y-a)+(-2)\left(z-\frac{\pi a}{4}\right)=0$$

$$-x+y-2z+a-a+\frac{\pi a}{\cancel{4}\,2}=0$$

$$-x+y-2z+\frac{\pi a}{2}=0$$

jednačina tangentne ravni helikoida u tački $(a,a,\frac{\pi a}{4})$.

$$d=\frac{Ax_1+By_1+Cz_1+D}{\pm\sqrt{A^2+B^2+C^2}}, \quad O(0,0,0)$$

$$d=\frac{0+0+0+\frac{\pi a}{2}}{\sqrt{1+1+4}}=\frac{\pi a}{2\sqrt{6}}$$

udaljenost početka koordinatnog sistema od tangentne ravni

Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}}+2^{\frac{y}{z}}=8$ u tački $M(2,2,1)$.

Rj. Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$ tada jednačina tangentne ravni i normale na površ S u tački $M(p_1,p_2,p_3)$ se računaju po formuli:

$$\alpha:\; F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

$$n:\; \frac{x-p_1}{F'_x(p_1,p_2,p_3)}=\frac{y-p_2}{F'_y(p_1,p_2,p_3)}=\frac{z-p_3}{F'_z(p_1,p_2,p_3)}$$

$$2^{\frac{x}{z}}+2^{\frac{y}{z}}=8 \qquad \left(\frac{x}{z}\right)'_z=\left(xz^{-1}\right)'_z=(-1)xz^{-2}$$

$$F(x,y,z)=2^{\frac{x}{z}}+2^{\frac{y}{z}}-8=0$$

$$F'_x=2^{\frac{x}{z}}\ln 2\cdot\frac{1}{z} \implies F'_x(2,2,1)=4\ln 2.$$

$$F'_y=2^{\frac{y}{z}}\ln 2\cdot\frac{1}{z} \implies F'_y(2,2,1)=4\ln 2$$

$$F'_z=2^{\frac{x}{z}}\ln 2\cdot\left(\frac{x}{z}\right)'_z+2^{\frac{y}{z}}\ln 2\left(\frac{y}{z}\right)'_z=-\frac{x}{z^2}2^{\frac{x}{z}}\ln 2-\frac{y}{z^2}2^{\frac{y}{z}}\ln 2$$

$$=-\frac{1}{z^2}\ln 2\left(x2^{\frac{x}{z}}+y2^{\frac{y}{z}}\right)$$

$$F'_z(2,2,1)=-\ln 2(2\cdot 4+2\cdot 4)=-16\ln 2$$

$$4\ln 2(x-2)+4\ln 2(y-2)+(-16\ln 2)(z-1)=0.$$

$$4x\ln 2+4y\ln 2-16z\ln 2+8\ln 2=0 \quad \text{jednačina tangentne ravni}$$

$$\frac{x-2}{4\ln 2}=\frac{y-2}{4\ln 2}=\frac{z-1}{-16\ln 2} \implies \frac{x-2}{1}=\frac{y-2}{1}=\frac{z-1}{-4}$$

jednačina normale na površ

Naći jednačinu tangentne ravni elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ koja na koordinatnim osama odsjeca jednake pozitivne odsječke.

Rj. Jednačina tangentne ravni na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$ ima jednačinu $F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)$

Nađimo jednačinu tangentne ravni na elipsoid u proizvoljnoj tački $M(p_1,p_2,p_3)$: (U našem slučaju $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$)

$$F'_x=\frac{1}{a^2}\cdot 2x=\frac{2x}{a^2},\quad F'_y=\frac{2y}{b^2},\quad F'_z=\frac{2z}{c^2}$$

$$F'_x(M)=\frac{2p_1}{a^2},\quad F'_y(M)=\frac{2p_2}{b^2},\quad F'_z(M)=\frac{2p_3}{c^2}$$

$$\frac{2p_1}{a^2}(x-p_1)+\frac{2p_2}{b^2}(y-p_2)+\frac{2p_3}{c^2}(z-p_3)=0 \quad /\cdot\frac{1}{2}$$

$$\frac{p_1}{a^2}x+\frac{p_2}{b^2}y+\frac{p_3}{c^2}z=\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}$$

Napišimo jednačinu ravni u kanonskom obliku

$$\frac{x}{\frac{a^2}{p_1}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{y}{\frac{b^2}{p_2}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{z}{\frac{c^2}{p_3}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}=1$$

Odavde možemo primjetiti da ako želimo da jednačina tangentne ravni na kordanatnim osama odsjeca jednake odsječke, potrebno i dovoljno je da $\frac{a^2}{p_1}=\frac{b^2}{p_2}$, $\frac{a^2}{p_1}=\frac{c^2}{p_3}$ i $\frac{b^2}{p_2}=\frac{c^2}{p_3}$. ...(*)

Isto tako primjetimo da je $\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}=1$ (ZAŠTO?) ...(**)

$$(*) \implies p_1=\frac{a^2}{b^2}p_2,\quad p_3=\frac{c^2}{b^2}p_2 \quad ...(1)$$

Sad imamo:

$$\frac{x}{\frac{a^2}{\frac{a^2}{b^2}p_2}}+\frac{y}{\frac{b^2}{p_2}}+\frac{z}{\frac{c^2}{\frac{c^2}{b^2}p_2}}=1 \quad /:p_2$$

$$\frac{x}{b^2}+\frac{y}{b^2}+\frac{z}{b^2}=\frac{1}{p_2}$$

Kada (1) stavimo u (**) dobićemo da je $p_2=\frac{b^2}{\sqrt{a^2+b^2+c^2}}$

Prema tome:

$x+y+z=\sqrt{a^2+b^2+c^2}$ je jednačina tražene tangente

Izvod f-je u datom pravcu (smjeru). Gradijent f-je.

Neka je data f-ja $u=f(x,y,z)$, diferencijabilna u oblasti $D\subseteq\mathbb{R}^3$ i $(p_1,p_2,p_3)\in D$.

Gradijentom f-je f u tački $M(p_1,p_2,p_3)$ naziva se vektor označen simbolom $\text{grad}\,u(M)$ koji ima koordinate

$$\text{grad}\,u(M)=\left(\frac{\partial u(M)}{\partial x},\ \frac{\partial u(M)}{\partial y},\ \frac{\partial u(M)}{\partial z}\right)=\frac{\partial u(M)}{\partial x}\vec{i}+\frac{\partial u(M)}{\partial y}\vec{j}+\frac{\partial u(M)}{\partial z}\vec{k}$$

Izvod f-je $u=f(x,y,z)$ u tački $M(p_1,p_2,p_3)$ u pravcu prave l (ili u smjeru prave l, u smjeru vektora $\vec{a}$ i slično) se računa po formuli

$$\frac{\partial u(M)}{\partial \vec{e}}=\text{grad}\,u(M)\cdot\vec{e},$$

gdje je $\vec{e}$ jedinični vektor

$$\vec{e}=\frac{\vec{a}}{|\vec{a}|}$$

Ako je vektor $\vec{e}$ zadan kosinusima $(\cos\alpha, \cos\beta, \cos\gamma)$, onda se izvod u pravcu vektora $\vec{e}$ nalazi po formuli

$$\frac{\partial u}{\partial \vec{e}}=\frac{\partial u}{\partial x}\cos\alpha+\frac{\partial u}{\partial y}\cos\beta+\frac{\partial u}{\partial z}\cos\gamma$$

Gradijent f-je $u=f(x,y,z)$ u tački M je vektor čije su projekcije (na ose dekartovog koordinatnog sistema) $f'_x(M)$, $f'_y(M)$ i $f'_z(M)$.

\# Izračunati izvod f-je $u=x^2y^2+z^2-3xyz$ u tački $T(1,1,2)$ u smjeru koji čini s koordinatnim osama uglove $\frac{\pi}{3}$, $\frac{\pi}{4}$ i $\frac{\pi}{6}$.

Rj: Izvod f-je $u=f(x,y,z)$ u tački $M(p_1,p_2,p_3)$ u pravcu vektora $\vec{e}$ ($\vec{e}$ je jedinični vektor) se računa po formuli:

$$\frac{\partial u(M)}{\partial \vec{e}}=\text{grad}\,u(M)\cdot\vec{e}$$

$$\frac{\partial u}{\partial x}=2xy^2-3yz$$
$$\frac{\partial u}{\partial y}=2x^2y-3xz$$
$$\frac{\partial u}{\partial z}=2z-3xy$$

$$\text{grad}\,u(M)=\left(\frac{\partial u}{\partial x}(M),\ \frac{\partial u}{\partial y}(M),\ \frac{\partial u}{\partial z}(M)\right)$$

$$\frac{\partial u}{\partial x}(1,1,2)=2\cdot1\cdot1-3\cdot1\cdot2=-4$$
$$\frac{\partial u}{\partial y}(1,1,2)=2\cdot1\cdot1-3\cdot1\cdot2=-4$$
$$\frac{\partial u}{\partial z}(1,1,2)=2\cdot2-3\cdot1\cdot1=1$$

$$\text{grad}\,u(M)=(-4,-4,1)$$

$$\vec{e}=\cos\alpha\,\vec{i}+\cos\beta\,\vec{j}+\cos\gamma\,\vec{k}=(\cos\alpha,\cos\beta,\cos\gamma)$$

$$\vec{e}=\left(\frac{1}{2},\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{3}}{2}\right) \qquad \alpha=\frac{\pi}{3},\ \beta=\frac{\pi}{4},\ \gamma=\frac{\pi}{6}$$

$$\frac{\partial u}{\partial \vec{e}}(M)=(-4,-4,1)\left(\frac{1}{2},\ \frac{\sqrt{2}}{2},\ \frac{\sqrt{3}}{2}\right)=-2-2\sqrt{2}+\frac{\sqrt{3}}{2}$$

$$|\vec{e}|=\sqrt{\frac{1}{4}+\frac{2}{4}+\frac{3}{4}}=1$$

$$\frac{\partial u}{\partial \vec{e}}(M)=\frac{\sqrt{3}}{2}-2\sqrt{2}-2$$

traženo rješenje (izvod f-je u u tački T u datom smjeru)

Naći izvod f-je $u=x^2-3yz+5$ u tački $T(1,2,-1)$ u smjeru koji čini jednake uglove sa svim koordinatnim osama.

Rj. Neka je data f-ja $u=f(x,y,z)$. Ako je vektor $\vec{e}$ zadan kosinusima $\vec{e}=(\cos\alpha, \cos\beta, \cos\gamma)$ onda se izvod u pravcu vektora $\vec{e}$ nalazi po formuli

$$\frac{\partial u}{\partial \vec{e}} = \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma .$$

$\vec{e}$ je jedinični vektor pa $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$

u našem slučaju je $\alpha=\beta=\gamma$ pa je

$$3\cos^2\alpha=1 \qquad \cos\alpha = \pm\frac{\sqrt{3}}{3}$$

$$\cos^2\alpha=\frac{1}{3}$$

$$\vec{e_1}=\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$$

$$\vec{e_2}=\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$$

$$\text{grad}\, u=\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$u'_x=2x \qquad u'_y=-3z \qquad u'_z=-3y$$

$$\text{grad}\, u(T)=(2,3,-6)$$

$$\frac{\partial u(T)}{\partial \vec{e_1}} = (2,3,-6)\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}+3\sqrt{3}-6\sqrt{3}}{3} = -\frac{\sqrt{3}}{3}$$

$$\frac{\partial u(T)}{\partial \vec{e_2}} = (2,3,-6)\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) = \frac{-2\sqrt{3}-3\sqrt{3}+6\sqrt{3}}{3} = \frac{\sqrt{3}}{3}$$

$$\frac{\partial u(T)}{\partial \vec{e_1}} = -\frac{\sqrt{3}}{3} \ ; \ \frac{\partial u(T)}{\partial \vec{e_2}} = \frac{\sqrt{3}}{3}$$

izvodi f-ja u tački T u smjeru koji čini jednake uglove sa svim koordinatnim osama

Naći izvod f-je $u=xyz$ u tački $M(1,1,1)$ u pravcu vektora $\vec{e}=(\cos\alpha, \cos\beta, \cos\gamma)$. Kolika je veličina gradijenta f-je u toj tački?

Rj.

$$\frac{\partial u(M)}{\partial \vec{e}} = \frac{\partial u(M)}{\partial x}\cos\alpha + \frac{\partial u(M)}{\partial y}\cos\beta + \frac{\partial u(M)}{\partial z}\cos\gamma$$

izvod f-je $u=xyz$ u tački M u pravcu vektora $\vec{e}=(\cos\alpha,\cos\beta,\cos\gamma)$

$$\frac{\partial u}{\partial x}=yz \Rightarrow u'_x(M)=1$$

$$\frac{\partial u}{\partial y}=xz \Rightarrow u'_y(M)=1$$

$$\frac{\partial u}{\partial z}=xy \Rightarrow u'_z(M)=1$$

$$\frac{\partial u}{\partial \vec{e}} = \cos\alpha+\cos\beta+\cos\gamma$$

izvod f-je u tački M u pravcu vektora $\vec{e}$

$$|\text{grad}\, u(M)| = \sqrt{\left(\frac{\partial u(M)}{\partial x}\right)^2+\left(\frac{\partial u(M)}{\partial y}\right)^2+\left(\frac{\partial u(M)}{\partial z}\right)^2} = \sqrt{1+1+1} = \sqrt{3}$$

veličina gradijenta f-je u tački M

Naći izvod f-je $z=x^2-y^2$ u tački $M(1,1)$ u pravcu vektora $\vec{e}$, koji sa pozitivnim dijelom x ose gradi ugao $\alpha=60°$.

Rj. Ako je $\vec{e}=(\cos\alpha,\sin\alpha)$ onda se izvod u pravcu vektora $\vec{e}$ nalazi po formuli

$$\frac{\partial z}{\partial \vec{e}} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha = \text{grad}\, z\cdot\vec{e}$$

$$\text{grad}\, z=\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \qquad \frac{\partial z}{\partial x}=2x \qquad \text{grad}\, z(M)=(2,-2)$$

$$\frac{\partial z}{\partial y}=-2y \qquad M(1,1)$$

$$\cos\alpha=\frac{1}{2}$$

$$\sin\alpha=\frac{\sqrt{3}}{2} \qquad \frac{\partial z(M)}{\partial \vec{e}} = 2\cdot\frac{1}{2}+(-2)\cdot\frac{\sqrt{3}}{2} = 1-\sqrt{3}$$

$$\frac{\partial z(M)}{\partial \vec{e}}=1-\sqrt{3}$$

izvod f-je z u tački M koji sa pozitivnim djelom x ose gradi ugao α

(#) Odrediti izvod f-je $u=x^2yz$ u tački $A(1,2,3)$ u pravcu prema tački $B(3,2,1)$.

Rj: Izvod f-je $u=f(x,y,z)$ u tački $A(p_1,p_2,p_3)$ u pravcu vektora $\vec{e}$ ($\vec{e}$ je jedinični vektor) računamo po formuli:

$$\frac{\partial u}{\partial \vec{e}} = \operatorname{grad} u(A)\cdot \vec{e}$$

$$u=x^2yz \qquad \frac{\partial u}{\partial x} = 2xyz$$

$$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) \qquad A(1,2,3) \qquad \frac{\partial u}{\partial y} = x^2 z$$

$$\frac{\partial u}{\partial z} = x^2 y$$

$$\operatorname{grad} u = \left(2xyz, x^2z, x^2y\right)$$

$$\operatorname{grad} u(A) = (12, 3, 2)$$

$A(1,2,3)$
$B(3,2,1)$

$$\overrightarrow{AB} = (2, 0, -2)$$

$$|\overrightarrow{AB}| = \sqrt{4+0+4} = \sqrt{8} = 2\sqrt{2}$$

$\overrightarrow{AB}$ nije jedinični vektor

od njega ćemo napraviti jedinični vektor

$$\vec{e} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{(2,0,-2)}{2\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial u(A)}{\partial \vec{e}} = (12,3,2)\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \frac{12}{\sqrt{2}} + 0 - \frac{2}{\sqrt{2}} = \frac{10}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

$\frac{\partial u(A)}{\partial \vec{e}} = 5\sqrt{2}$ izvod f-je u datom smjeru u tački A

(#) Izračunati izvod f-je $z=\operatorname{arctg} xy$ u tački $A(1,1)$ u smjeru simetrale prvog kvadranta.

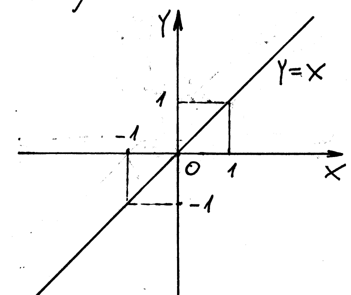
Rj: Nađimo gradijent f-je z

$$\operatorname{grad} z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) \qquad z=\operatorname{arctg} xy$$

$$\frac{\partial z}{\partial x} = \frac{y}{1+x^2y^2}$$

$$\frac{\partial z}{\partial y} = \frac{x}{1+x^2y^2} \qquad \operatorname{grad} z(A) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Simetrala prvog kvadranta je prava čije su tačke jednako udaljene od x i y ose. To je prava $y=x$.



Izvod f-je $z=f(x,y)$ u tačk $A(p_1,p_2)$ u smjeru vektora $\vec{e}$ ($\vec{e}$ je jedinični vektor) računamo po formuli

$$\frac{\partial z}{\partial \vec{e}} = \operatorname{grad} z(A)\cdot \vec{e}$$

$A(1,1)$

$\overrightarrow{OA} = (1,1)$

$$|\overrightarrow{OA}| = \sqrt{1+1} = \sqrt{2}$$

$$\vec{e} = \frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial z(A)}{\partial \vec{e}} = \left(\frac{1}{2}, \frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\frac{\partial z(A)}{\partial \vec{e}} = \frac{\sqrt{2}}{2}$ izvod f-je z u datom smjeru u tački A

Izračunati ugao između gradijenta f-je $u=\frac{x}{x^2+y^2+z^2}$ u tački $A(1,2,2)$ i gradijenta te iste f-je u tački $B(-3,1,0)$.

Rj.

$$\operatorname{grad} u=\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)$$

$$\frac{\partial u}{\partial x}=\frac{1\cdot(x^2+y^2+z^2)-x\cdot 2x}{(x^2+y^2+z^2)^2}=\frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial u}{\partial y}=\frac{0\cdot(x^2+y^2+z^2)-x\cdot 2y}{(x^2+y^2+z^2)^2}=\frac{-2xy}{(x^2+y^2+z^2)^2}$$

na isti način

$$\frac{\partial u}{\partial z}=\frac{-2xz}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial u}{\partial x}(1,2,2)=\frac{-1+4+4}{(1+\ +4)^2}=\frac{7}{81}\qquad \frac{\partial u}{\partial x}(-3,1,0)=\frac{-9+1+0}{(9+1+0)^2}=\frac{-8}{100}$$

$$\frac{\partial u}{\partial y}(1,2,2)=\frac{-2\cdot 1\cdot 2}{(1+4+4)^2}=-\frac{4}{81}\qquad \frac{\partial u}{\partial y}(-3,1,0)=\frac{6\cdot 1}{10^2}=\frac{6}{100}$$

$$\frac{\partial u}{\partial z}(1,2,2)=\frac{-2\cdot 1\cdot 2}{(1+4+4)^2}=-\frac{4}{81}\qquad \frac{\partial u}{\partial z}(-3,1,0)=\frac{0}{100}=0$$

$$\vec{a}=\operatorname{grad} z(A)=\left(\frac{7}{81},-\frac{4}{81},-\frac{4}{81}\right)=\frac{7}{81}\vec{i}-\frac{4}{81}\vec{j}-\frac{4}{81}\vec{k}$$

$$\vec{b}=\operatorname{grad} z(B)=\left(-\frac{8}{100},\frac{6}{100},0\right)=-\frac{8}{100}\vec{i}+\frac{6}{100}\vec{j}$$

Nađimo ugao između ova dva vektora

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\measuredangle(\vec{a},\vec{b})$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$|\vec{a}|=\sqrt{\frac{49+16+16}{81^2}}=\sqrt{\frac{81}{81^2}}=\frac{1}{9}$$

$$|\vec{b}|=\sqrt{\frac{64+36}{100^2}}=\sqrt{\frac{100}{100^2}}=\frac{1}{10}$$

$$\vec{a}\cdot\vec{b}=\left(\frac{7}{81},-\frac{4}{81},-\frac{4}{81}\right)\left(-\frac{8}{100},\frac{6}{100},0\right)=\frac{-7\cdot 8}{81\cdot 100}-\frac{4\cdot 6}{81\cdot 100}=\frac{-56-24}{81\cdot 100}=\frac{-80}{81\cdot 100}=\frac{-8}{810}$$

$$\vec{a}\cdot\vec{b}=\frac{-4}{405}=\frac{-4}{5\cdot 81}$$

$$\cos\measuredangle(\vec{a},\vec{b})=\frac{-\frac{4}{5\cdot 81}}{\frac{1}{9}\cdot\frac{1}{10}}=\frac{-4\cdot\cancel{9}^{1}\cdot\cancel{10}^{2}}{\cancel{5}_{1}\cdot\cancel{81}_{9}}=-\frac{8}{9}$$

$$\varphi=\arccos\left(-\frac{8}{9}\right)$$ ugao između naša dva gradijenta

Izračunati izvod f-je $z=\operatorname{arctg}\frac{y}{x}$ u tački $T\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ (koja leži na kružnici $x^2+y^2-2x=0$) u smjeru te kružnice.

Rj.

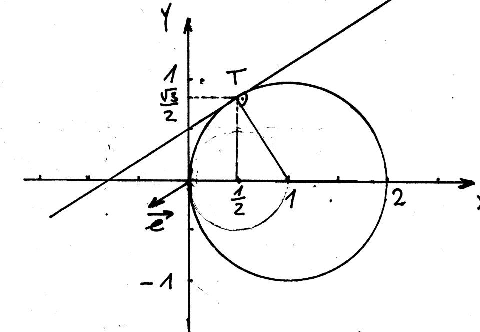
$$x^2-2x+y^2=0$$

$$x^2-2\cdot 1\cdot x+1-1+y^2=0$$

$$(x-1)^2+y^2=1$$

$S(1,0)$ centar kružnice

$r=1$ poluprečnik



u smjeru kružnice, misli se na smjer tangente na kružnicu u datoj tački

$$x^2-2x+y^2=0\quad /\frac{d}{dx}$$

$$2x-2+2yy'=0$$

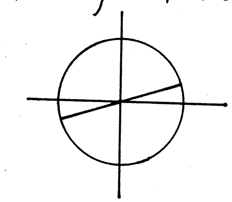
$$2yy'=-2x+2\quad /:y\ /:2$$

$$y'=\frac{1-x}{y}$$

$y'(T)=k=\operatorname{tg}\alpha$ koeficijent pravca tangente u tački T

α – ugao koji tangenta zatvara sa pozitivnim dijelom x-ose

$$T\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)\qquad y'(T)=\frac{1-\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$



$$\operatorname{tg}\alpha=\frac{\sqrt{3}}{3}\Rightarrow\alpha=\frac{\pi}{6}\quad\text{ili}\quad\alpha=\frac{7\pi}{6}$$

$\vec{e}$ jedinični vektor koji kreće iz koordinatnog početka i paralelan je sa tangentom

$$\alpha=\frac{7\pi}{6},\quad\vec{e}=\left(\cos\frac{7\pi}{6},\sin\frac{7\pi}{6}\right)=\left(-\cos\frac{\pi}{6},-\sin\frac{\pi}{6}\right)=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$$

$$\operatorname{grad} z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right),\quad \frac{\partial z}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)=\frac{-y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$

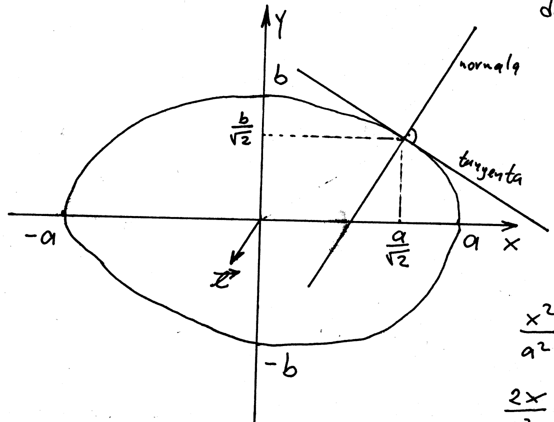
$$\operatorname{grad} z(T)=\left(\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{4}+\frac{3}{4}},\frac{\frac{1}{2}}{\frac{1}{4}+\frac{3}{4}}\right)$$

$$\frac{\partial z(T)}{\partial\vec{e}}=\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)\cdot\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)=\frac{3}{4}-\frac{1}{4}=\frac{2}{4}=\frac{1}{2}$$

izvod f-je z u tački T u smjeru kružnice

\# Nađi izvod f-je $z=1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$ u tački $M\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$ u smjeru unutrašnje normale u toj tački na krivu $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Rj. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ jednačina elipse, tačka $M\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$ pripada datoj elipsi



normala u datuj tačku krive je prava koja je okomita na tangentu povučena u toj tački krive

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \Big/ \frac{d}{dx}$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0, \quad M\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$$

$$\frac{2\cdot\frac{a}{\sqrt{2}}}{a^2}+\frac{2\frac{b}{\sqrt{2}}}{b^2}y'=0$$

$k=y'(M)=\operatorname{tg}\alpha$ koeficijent pravca tangente na krivu u tački M

$k_N\cdot k_T=-1$

$\operatorname{tg}\varphi=\frac{a}{b}$ koeficijent pravca normale na krivu u tački M

$$\frac{2}{b\sqrt{2}}y'=\frac{-2}{a\sqrt{2}}$$

$$y'=-\frac{b}{a}$$

$\vec{e}$ je jedinični vektor čiji je početak u $(0,0)$ i koji je paralelan sa normalom

$y-y_1=k(x-x_1)$

$(x_1,y_1)=(0,0)$

$y=\frac{a}{b}x$

$\vec{e}=\left(x,\frac{a}{b}x\right)$, $x=?$ $\quad |\vec{e}|=1$ $\quad \sqrt{x^2+\frac{a^2}{b^2}x^2}=1$ $\quad -x\frac{\sqrt{a^2+b^2}}{b}=1$ $\quad x=\frac{-b}{\sqrt{a^2+b^2}}$

$$\vec{e}=\left(\frac{-b}{\sqrt{a^2+b^2}},\frac{-a}{\sqrt{a^2+b^2}}\right)$$

$$\operatorname{grad} z=\left(\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)$$

$$\frac{\partial z}{\partial x}=-\frac{2x}{a^2}$$

$$\frac{\partial z}{\partial y}=\frac{-2y}{b^2}$$

$$\operatorname{grad} z(M)=\left(-\frac{\sqrt{2}}{a},-\frac{\sqrt{2}}{b}\right)$$

$$\frac{\partial z(M)}{\partial \vec{e}}=\operatorname{grad} z(M)\cdot\vec{e}=\frac{b\sqrt{2}}{a\sqrt{a^2+b^2}}+\frac{a\sqrt{2}}{b\sqrt{a^2+b^2}}$$

izvod f-je z u tački M u datom smjeru

Zadaci za vježbu

PRIMENA DIFERENCIJALNOG RAČUNA FUNKCIJA VIŠE PROMENLJIVIH

§ 1. Tajlorova formula. Ekstremumi funkcija više promenljivih

Tajlorova formula

3242. $f(x, y)=x^3+2y^3-xy$; razviti funkciju $f(x+h, y+k)$ po stepenima od h i k.

3243. $f(x, y)=x^3+y^3-6xy-39x+18y+4$; naći priraštaj koji dobija funkcija kad nezavisno promenljive, polazeći od vrednosti $x=5$, $y=6$, pređu na vrednosti $x=5+h$, $y=6+k$.

3244. $f(x, y)=\frac{xy^3}{4}-yx^3+\frac{x^2y^2}{2}-2x+3y-4$; naći priraštaj koji dobija funkcija kad nezavisno promenljive, polazeći od vrednosti $x=1$, $y=2$, pređu na vrednosti $x=1+h$, $y=2+k$. Zadržavajući članove do drugog stepena zaključno — izračunati $f(1{,}02;\ 2{,}03)$.

3245. $f(x, y, z)=Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx$; razviti $f(x+h, y+k, z+l)$ po stepenima od h, k i l.

3246. Razviti $z=\sin x\sin y$ po stepenima razlika $\left(x-\frac{\pi}{4}\right)$ i $\left(y-\frac{\pi}{4}\right)$; naći članove prvog i drugog stepena i R_2 (ostatak drugoga reda).

3247. Funkciju $z=x^y$ razviti po stepenima razlika $(x-1)$ i $(y-1)$ idući do članova trećeg stepena zaključno. Koristeći dobijeni rezultat izračunati $1{,}1^{1{,}02}$ (bez upotrebe tablica).

3248. $f(x, y)=e^x\sin y$; razviti $f(x+h, y+k)$ po stepenima od h i k zadržavajući članove do trećeg stepena po h i k zaključno. Koristeći dobijeni rezultat izračunati $e^{0{,}1}\sin 0{,}49\pi$.

3249. Naći nekoliko prvih članova Tajlorovog reda za funkciju $e^x\sin y$ razvijenu u okolini tačke $(0,0)$.

3250. Naći nekoliko prvih članova Tajlorovog reda za funkciju $e^x\ln(1+y)$ razvijenu u okolini tačke $(0,0)$.

U zadacima 3251 — 3256 date funkcije razviti u Tajlorov red za $x_0=0$, $y_0=0$.

3251. $z=\frac{1}{1-x-y+xy}$. **3252*.** $z=\operatorname{arctg}\frac{x-y}{1+xy}$.

3253. $z=\ln(1-x)\ln(1-y)$.

3254. $z=\ln\frac{1-x-y+xy}{1-x-y}$. **3255.** $z=\sin(x^2+y^2)$.

3256. $z=e^x\cos y$.

3257. Funkciju z definisanu implicitno jednačinom

$$z^3-yz-xy^2-x^3=0$$

za $x\neq 1$ i $y\neq 1$, i vrednošću $z=1$ za $x=y=1$, razviti u red po stepenima razlika $x-1$ i $y-1$ i naći nekoliko prvih članova toga reda.

3258. Izvesti približnu formulu

$$\frac{\cos x}{\cos y}\approx 1-\frac{1}{2}(x^2-y^2)$$

za dovoljno male vrednosti $|x|$ i $|y|$.

Lokalne ekstremne vrednosti

U zadacima 3259 — 3267 naći stacionarne tačke datih funkcija.

3259. $z=2x^3+xy^2+5x^2+y^2$. **3260.** $z=e^{2x}(x+y^2+2y)$.

3261. $z=xy(a-x-y)$. **3262.** $z=(2ax-x^2)(2by-y^2)$.

3263. $z=\sin x+\sin y+\cos(x+y)$ $\left(0<x<\frac{\pi}{4},\ 0<y<\frac{\pi}{4}\right)$.

3264. $z=\frac{a+bx+cy}{\sqrt{1+x^2+y^2}}$ **3265.** $z=y\sqrt{1+x}+x\sqrt{1+y}$.

3266. $u=2x^2+y^2+2z-xy-xz$.

3267. $u=3\ln x+2\ln y+5\ln z+\ln(22-z-y-z)$.

3268. Na sl. 60 predstavljene su nivoske linije funkcije $z=f(x, y)$. Kakve osobenosti pokazuje ova funkcija u Tačkama A, B, C, D, i na pravoj EF?

3269. Funkcija z definisana je implicitno jednačinom

$$2x^2+2y^2+z^2+8xz-z+8=0.$$

Naći njene stacionarne tačke.

3270. Funkcija z definisana je implicitno jednačinom

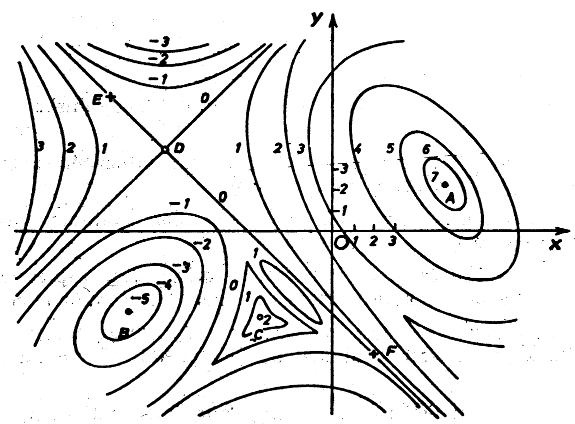
$$5x^2+5y^2+5z^2-2xy-2xz-2yz-72=0.$$

Naći njene stacionarne tačke.

3271*. Naći. tačke ekstremuma funkcije

$$z=2xy-3x^2-2y^2+10.$$

3272. Naći tačke ekstremuma funkcije $z=4(x-y)-x^2-y^2$.



Sl. 60

3273. Naći tačke ekstremuma funkcije $z=x^2+xy+y^2+x-y+1$.

3274. Uveriti se da funkcija $z=x^2+xy+y^2+\frac{a^3}{x}+\frac{a^3}{y}$ ima minimum u tački $x=y=\frac{a}{\sqrt{3}}$.

3275. Uveriti se da za $x=\sqrt{2}$, $y=\sqrt{2}$ i za $x=-\sqrt{2}$, $y=-\sqrt{2}$ funkcija $z=x^4+y^4-2x^2-4xy-2y^2$ ima minimum.

3276. Uveriti se da za $x=5$, $y=6$ funkcija $z=x^3+y^2-6xy-39x+18y+20$ ima minimum.

3277. Naći stacionarne tačke funkcije $z=x^3y^2(12-x-y)$, koje zadovoljavaju uslov $x>0$, $y>0$ i ispitati njihov karakter.

3278. Naći stacionarne tačke funkcije $z=x^3+y^3-3xy$ i ispitati njihov karakter.

Ekstremne vrednosti u datoj oblasti

3279. Naći najveću i najmanju vrednost funkcije $z=x^2-y^2$ u krugu $x^2+y^2<4$.

3280. Naći najveću i najmanju vrednost funkcije $z=x^2+2xy-4x+8y$ u pravougaoniku $0<x<1$, $0<y<2$.

3281. Naći najveću vrednost funkcije $z=x^2y(4-x-y)$ u trouglu koji obrazuju prave $x=0$, $y=0$, $x+y=6$.

3282. Naći najveću i najmanju vrednost funkcije $z=e^{-x^2-y^2}(2x^2+3y^2)$ u krugu $x^2+y^2<4$.

3283. Naći najveću i najmanju vrednost funkcije

$$z=\sin x+\sin y+\sin(x+y)$$

u pravougaoniku $0<x<\frac{\pi}{2}$; $0<y<\frac{\pi}{2}$.

3284. Pozitivan broj a razložiti na tri proizvoljna sabirka tako da njihov proizvod bude minimalan.

3285. Pozitivan broj a predstaviti u obliku proizvoda četiri pozitivna množitelja tako da njihov zbir bude minimalan.

3286. U ravni Oxy naći tačku za koju je zbir kvadrata odstojanja od pravih $x=0$, $y=0$, $x+2y-16=0$ minimalan.

3287. Kroz tačku (a, b, c) postaviti ravan tako da zapremina tetraedra koji ta ravan obrazuje sa koordinatnim ravnima, bude minimalna.

3288. Date su tačke $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$; u ravni Oxy naći tačku za koju će zbir kvadrata odstojanja od svih datih tačaka biti minimalan.

3289. Date su tri tačke $A(0, 0, 12)$, $B(0, 0, 4)$ i $C(8, 0, 8)$; u ravni Oxy naći tačku D tako da poluprečnik sfere koja prolazi kroz tačke $ABCD$ bude minimalan.

3290. U loptu prečnika $2R$ upisati pravougli paralelepiped maksimalne zapremine.

Uslovne ekstremne vrednosti

U zadacima 3291 — 3296 naći ekstremne vrednosti funkcija.

3291. $z=x^m+y^m\ (m>1)$ za $x+y=2$ $(x>0,\ y>0)$.

3292. $z=xy$ za $x^2+y^2=2a^2$.

3293. $z=\frac{1}{x}+\frac{1}{y}$ za $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{a^2}$.

3294. $z=a\cos^2x+b\cos^2y$ za $y-x=\frac{\pi}{4}$.

3295. $u=x+y+z$ za $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

3296. $u = xyz$ za $\begin{cases} 1) & x+y+z=5, \\ 2) & xy+xz+yz=8. \end{cases}$

3297*. Dokazati da važi nejednakost

$$\frac{x_1^2+x_2^2+\cdots+x_n^2}{n} > \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^2.$$

3298. $f(x, y) = x^3 - 3xy^2 + 18y$, pri čemu je $3x^2y - y^3 - 6x = 0$. Dokazati da funkcija $f(x, y)$ dostiže ekstremum u tačkama $x = y = \pm\sqrt{3}$.

3299. Naći minimum funkcije $u = ax^2 + by^2 + cz^2$, pri čemu su a, b, c pozitivne konstante, a x, y, z su vezani realizacijom $x + y + z = 1$.

3300. Naći najveću i najmanju vrednost funkcije

$$u = y^2 + 4z^2 - 4yz - 2xz - 2xy$$

pod uslovom $2x^2 + 3y^2 + 6z^2 = 1$.

3301. U ravni $3x - 2z = 0$ naći tačku za koju je zbir kvadrata odstojanja od $A(1, 1, 1)$ i $B(2, 3, 4)$ minimalan.

3302. U ravni $x + y - 2z = 0$ naći tačku za koju je zbir kvadrata odstojanja od ravni $x + 3z = 6$ i $y + 3z = 2$ minimalan.

3303. Date su tačke $A(4, 0, 4)$, $B(4, 4, 4)$, $C(4, 4, 0)$. Na površini lopte $x^2 + y^2 + z^2 = 4$ naći tačku S tako da zapremina piramide $SABC$ bude: a) maksimalna, b) minimalna. Proveriti tačnost rezultata metodama elementarne geometrije.

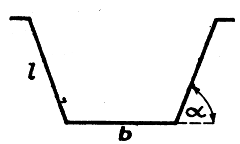
3304. Naći pravougli paralelepiped date zapremine V čija je površina minimalna.

3305. Naći pravougli paralelepiped date površine S čija je zapremina maksimalna.

3306. Naći zapreminu najvećeg pravouglog paralelepipeda koji se može upisati u elipsoid sa poluosama a, b i c.

3307. Šator date zapremine ima oblik cilindra sa konusnim završetkom. U kom odnosu moraju stajati dimenzije šatora da bi količina materijala, potrebnog za njegovu izradu, bila minimalna?

3308. Presek kanala ima oblik jednakokrakog trapeza date površine; kolike moraju biti njegove dimenzije da bi kvašena površina kanala bila najmanja? (sl. 61)



Sl. 61

3309. Od svih pravouglih paralelepipeda koji imaju datu dijagonalu naći onaj čija je zapremina maksimalna.

3310. Odrediti spoljne dimenzije otvorenog (bez poklopca) sanduka koji ima oblik pravouglog paralelepipeda sa datom debljinom zidova α i datom zapreminom V, tako da bi količina materijala potrebnog za njegovu izradu bila minimalna.

3311. Odrediti paralelepiped najveće zapremine čiji zbir svih 12 ivica ima datu vrednost ($12a$).

3312. Oko date elipse opisati trougao najmanje površine, čija je osnovica paralelna velikoj osi elipse.

3313. Na elipsi $\frac{x^2}{4} + \frac{y^2}{9} = 1$ naći tačku čije je odstojanje od prave $3x - y - 9 = 0$ minimalno, odnosno maksimalno.

3314. Na paraboli $x^2 + 2xy + y^2 + 4y = 0$ naći tačku najbližu pravoj $3x - 6y + 4 = 0$.

3315. Na paraboli $2x^2 - 4xy + 2y^2 - x - y = 0$ naći tačku najbližu pravoj $9x - 7y + 16 = 0$.

3316. Naći maksimalno odstojanje tačaka površine

$$2x^2 + 3y^2 + 2z^2 = 6$$

od ravni $z = 0$.

3317. Naći stranice pravouglog trougla date površine S čiji je obim minimalan.

3318. U prav eliptični konus čije su poluose osnove a i b cm, a visina H cm, upisana je prizma sa pravougaonom osnovom tako da su osnovne ivice paralelne osama elipse, a presek dijagonala osnove leži u centru elipse. Kolike moraju biti osnovne ivice i visina prizme da bi njena zapremina bila maksimalna, i kolika je ta maksimalna zapremina?

3319. Naći pravilnu trostranu piramidu date zapremine, čiji je zbir svih ivica minimalan.

3320. Date su dve tačke elipse; odrediti položaj treće tačke elipse tako da površina trougla čija su temena pomenute tačke — bude maksimalna.

3321. Odrediti onu normalu elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ čije je odstojanje od koordinatnog početka maksimalno.

3322. Na obrtnom elipsoidu $\frac{x^2}{96} + y^2 + z^2 = 1$ naći tačku čije je odstojanje od ravni $3x + 4y + 12z = 288$ minimalno, odnosno maksimalno.

3323. Date su ravne krive $f(x, y) = 0$ i $\varphi(x, y) = 0$. Pokazati da će rastojanje između tačaka (α, β) i (ξ, η), od kojih prva leži na prvoj a druga na drugoj krivoj, imati ekstremnu vrednost ako su ispunjeni sledeći uslovi.

$$\frac{\alpha-\xi}{\beta-\eta} = \frac{\left(\frac{\partial f}{\partial x}\right)_{\substack{x=\alpha\\y=\beta}}}{\left(\frac{\partial f}{\partial y}\right)_{\substack{x=\alpha\\y=\beta}}} = \frac{\left(\frac{\partial \varphi}{\partial x}\right)_{\substack{x=\xi\\y=\eta}}}{\left(\frac{\partial \varphi}{\partial y}\right)_{\substack{x=\xi\\y=\eta}}}.$$

Koristeći se ovim rezultatom naći najkraće rastojanje između elipse $x^2 + 2xy + 5y^2 - 16y = 0$ i prave $x + y - 8 = 0$.

§ 4. Skalarno polje. Gradijent. Izvod u određenom pravcu

Gradijent

3439. 1) $\psi(x, y) = x^2 - 2xy + 3y - 1$. Naći komponente gradijenta u tački $(1, 2)$.

2) $u = 5x^2y - 3xy^3 + y^4$. Naći komponente gradijenta u proizvoljnoj tački.

3440. 1) $z = x^2 + y^2$. Naći grad z u tački $(3, 2)$.

2) $z = \sqrt{4 + x^2 + y^2}$. Naći grad z u tački $(2, 1)$.

3) $z = \operatorname{arctg} \frac{y}{x}$. Naći grad z u tački (x_0, y_0).

3441. 1) Naći najveći uspon (nagib) površi $z = \ln(x^2 + 4y^2)$ u tački $(6, 4, \ln 100)$.

2) Naći najveći uspon površi $z = x^y$ u tački $(2, 2, 4)$.

3442. Odrediti pravac najbržeg menjanja funkcije $\varphi(x, y, z) = x \sin z - y \cos z$ u koordinatnom početku.

3443. 1) $z = \arcsin \dfrac{x}{x+y}$. Naći ugao između gradijenata ove funkcije u tačkama (1, 1) i (3, 4).

2) Date su funkcije $z = \sqrt{x^2+y^2}$ i $z = x - 3y + \sqrt{3xy}$ Naći ugao između gradijenata tih funkcija u tački (3, 4).

3444. 1) Neka je $z = \ln\left(x + \dfrac{1}{y}\right)$; naći tačku u kojoj je grad $z = \boldsymbol{r} - \dfrac{16}{9}\boldsymbol{j}$.

2) Neka je $z = (x^2+y^2)^{\frac{3}{2}}$; naći tačku u kojoj je $|\text{grad } z| = 2$.

3445. Dokazati sledeće relacije (φ i ψ su diferencijabilne funkcije, c je konstanta):

$$\text{grad } (\varphi + \psi) = \text{grad } \varphi + \text{grad } \psi; \qquad \text{grad } (c + \varphi) = \text{grad } \varphi;$$

$$\text{grad } (c\varphi) = c \text{ grad } \varphi; \qquad \text{grad } (\varphi \psi) = \varphi \text{ grag } \psi + \psi \text{ grad } \varphi$$

$$\text{grad } (\varphi^n) = n\varphi^{n-1} \text{ grad } \varphi; \qquad \text{grad } [\varphi(\psi)] = \varphi'(\psi) \text{ grad } \psi.$$

3446. $z = \varphi(u, v)$, $u = \psi(x, y)$, $v = \xi(x, y)$. Pokazati da je

$$\text{grad } z = \frac{\partial \varphi}{\partial u} \text{ grad } u + \frac{\partial \varphi}{\partial v} \text{ grad } v.$$

3447. 1) $u(x, y, z) = x^3 y^2 z$. Naći knmpɔnente vektora grad u u tački (x_0, y_0, z).

2) $u(x, y, z) = \sqrt{x^2+y^2+z^2}$. Naći grad u.

3448. Pokazati da funkcija $u = \ln(x^2+y^2+z^2)$ zadovoljava relaciju $u = 2 \ln 2 - \ln (\text{grad } u)^2$.

3449. Dokazati da: ako su x, y, z funkcije nezavisno promenljive t onda je

$$\frac{d}{dt} f(x, y, z) = \text{grad } f \cdot \frac{d\boldsymbol{r}}{dt},$$

pri čemu je

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}.$$

3450. Koristeći obrazac izveden u prethodnom zadatku naći gradijent funkcije:

1) $f = r^2$; 2) $f = |\boldsymbol{r}|$; 3) $f = F(r^2)$; 4) $f = (\boldsymbol{ar})(\boldsymbol{br})$; $f = (\boldsymbol{abr})$; pri čemu su $\boldsymbol{a}$ i $\boldsymbol{b}$ konstantni vektori.

Izvod u određenom pravcu

3451. 1) Naći izvod funkcije $z = x^3 - 3x^2y + 3xy + 1$ u tački $M(3, 1)$ u pravcu vektora $\overrightarrow{MP}$, ako je $P(6, 5)$.

2) Naći izvod funkcije $z = \text{arctg } xy$ u tački (1, 1) u pravcu simetrale prvog kvadrata.

3) Naći izvod funkcije $z = x^2y^2 - xy^3 - 3y - 1$ u tački (2, 1) u pravcu koji vodi prema koordioatncm početku.

4) Naći izvod funkcije $z = \ln(e^x + e^y)$ u kordinatnom početku u pravcu određenom uglom α prema x-osi.

3452. Naći izvod funkcije $z = \ln(x+y)$ u tački (1, 2) parabole $y^2 = 4x$ u pravcu te parabole.

3453. Naći izvod funkcije $z = \text{arctg } \dfrac{y}{x}$ u tački $\left(\dfrac{1}{2}; \dfrac{\sqrt{3}}{2}\right)$ koja leži na krugu $x^2+y^2-2x=0$ u pravcu tog kruga.

3454. Dokazati da izvod funkcije $z = \dfrac{y^2}{x}$ u svakoj tački elipse $2x^2+y^2=1$, u pravcu normale na elipsu, ima vrednost nulu.

3455. 1) Naći izvod funkcije $u = xy^2 + z^3 - xyz$ u tački $M(1, 1, 2)$ u pravcu vektora $\overrightarrow{MP}$ koji sa koordinatnim osama zaklapa uglove od 60°, 45° i 60°

2) naći izvod funkcije $w = xyz$ u tački $A(5, 1, 2)$ u pravcu vektora $\overrightarrow{AB}$ pri čemu je $B(9, 4, 14)$.

3456. Naći izvod funkcije $u = x^2y^2z^2$ u tački $A(1, -1, 3)$ u pravcu koji vodi prema tački $B(0, 1, 1)$.

3457. Dokazati da izvod funkcije $u = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$ u proizvoljnoj tački $M(x, y, z)$ u pravcu koji od te tačke vodi prema koordinatnom početku, ima vrednost $-\dfrac{2u}{r}$, pri čemu je $r = \sqrt{x^2+y^2+z^2}$.

3458. Dokazati da je izvod funkcije $u = f(x, y, z)$ u pravcu njenog gradijenta jednak modulu tog gradijenta.

3459. Naći izvod funkcije

$$u = \frac{1}{r}, \text{ gde je } r^2 = x^2+y^2+z^2$$

u pravcu njenog gradijenta.

Rješenja

3242. $x^3+2y^3-xy-h(3x^2-y)+k(6y^2-x)+3xh^2-hk+6yk^2+h^3+2k^3$.

3243. $\Delta z=15h^2-6hk+k^2+h^3$.

3244. $\Delta z=-2h+7k-4h^2+4hk+2k^2-2h^3-h^2k+\frac{5}{2}hk^2+\frac{1}{4}k^3-h^3k+$

$$+\frac{1}{2}h^2k^2+\frac{1}{4}hk^3;\quad f(1{,}02;\ 2{,}03)\approx 2{,}1726.$$

3245. $Ax^2+By^2+Cz^2+Dxy+Eyz+Fzx+(2Ax+Dy+Fz)h+(2By+Dx+Ez)k+$

$$+(2Cz+Ey+Fx)l+Ah^2+Bk^2+Cl^2+Dhk+Ekl+Fhl.$$

3246. $z=\frac{1}{2}+\frac{1}{2}\left(x-\frac{\pi}{4}\right)+\frac{1}{2}\left(y-\frac{\pi}{4}\right)-\frac{1}{4}\left[\left(x-\frac{\pi}{4}\right)^2-\right.$

$$\left.-2\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)+\left(x-\frac{\pi}{4}\right)^2\right]-\frac{1}{6}\left[\cos\xi\cos\eta\left(x-\frac{\pi}{4}\right)^3+\right.$$

$$+3\sin\xi\cos\eta\left(x-\frac{\pi}{4}\right)^2\left(y-\frac{\pi}{4}\right)+3\cos\xi\sin\eta\left(x-\frac{\pi}{4}\right)\left(y-\frac{\pi}{4}\right)^2+$$

$$\left.+\sin\xi\cos\eta\left(y-\frac{\pi}{4}\right)^3\right].$$

3247. $z=1+(x-1)+(x-1)(y-1)+\frac{1}{2}(x-1)^2(y-1)+\cdots;\quad z\approx 1{,}1021.$

3248. $e^x\left[\sin y+h\sin y+k\cos y+\frac{1}{2}(h^2\sin y+2hk\cos y-k^2\sin y)+\right.$

$$\left.+\frac{1}{6}(h^3\sin y+3h^2k\cos y-3hk^2\sin y-k^3\cos y)\right]+\cdots;\quad z_1\approx 1{,}1051.$$

3249. $y+xy+\frac{1}{2}x^2y-\frac{1}{6}y^3+\cdots$

3250. $y+\frac{1}{2!}(2xy-y^2)+\frac{1}{3!}(3x^2y-3xy^2+2y^3)+\cdots$

3251. $1+(x+y)+\cdots+\frac{x^{n+1}-y^{n+1}}{x-y}+\cdots$

3252*. $x-y-\frac{1}{3}(x^3-y^3)+\frac{1}{5}(x^5-y^5)-\cdots+\frac{(-1)^n}{2n+1}(x^{2n+1}-y^{2n+1})+\cdots$ uzeti u obzir da je $\operatorname{arctg}\frac{x-y}{1+xy}=\operatorname{arctg}x-\operatorname{arctg}y$.

3253. $\left(\sum_{n=1}^{\infty}\frac{x^n}{n}\right)\left(\sum_{n=1}^{\infty}\frac{y^n}{n}\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^ny^m}{mn}$.

3254. $\sum_{n=2}^{\infty}\frac{(x+y)^n-x^n-y^n}{n}$. **3255.** $\sum_{n=0}^{\infty}(-1)^n\frac{(x^2+y^2)^{2n+1}}{(2n+1)!}$.

3256. $\sum_{m=0}^{\infty}\frac{x^m}{m!}\sum_{n=0}^{\infty}(-1)^n\frac{y^{2n}}{(2n)!}=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n\frac{x^my^{2n}}{m!\,(2n)!}$.

3257. $z=1+(x-1)+\frac{1}{4}(y-1)-\frac{1}{8}(x-1)(y-1)+\frac{9}{64}(y-1)^2+\cdots$

3259. $(0,\ 0)$, $\left(-\frac{5}{3},\ 0\right)$, $(-1,\ 2)$, $(-1,\ -2)$.

3260. $\left(\frac{1}{2},\ -1\right)$. **3261.** $(0,\ 0)$, $(0,\ a)$, $(a,\ 0)$, $\left(\frac{a}{3},\ \frac{a}{3}\right)$.

3262. $(0,\ 0)$, $(0,\ 2b)$, $(a,\ b)$, $(2a,\ 0)$, $(2a,\ 2b)$. **3263.** $\left(\frac{\pi}{6},\ \frac{\pi}{6}\right)$.

3264. $\left(\frac{b}{a},\ \frac{c}{a}\right)$. **3265.** $\left(-\frac{2}{3},\ -\frac{2}{3}\right)$. **3266.** $(2,\ 1,\ 7)$. **3267.** $(6,\ 4,\ 10)$.

3268. A i C su tačke maksimuma, B — tačka minimuma; u okolini tačke D površ ima oblik „sedla", duž prave EF funkcija zadržava konstantnu vrednost.

3269. $(-2,\ 0)$, $\left(\frac{16}{7},\ 0\right)$. **3270.** $(1,\ 1)$, $(-1,\ -1)$.

3271*. Da bismo se uverili da je nađena tačka — tačka maksimuma dovoljno je predstaviti funkciju u obliku $z=10-(x-y)^2-2x^2-y^2$.

3272. $(2,\ -2)$. **3273.** $(-1,\ 1)$. **3277.** U tački $(6,4)$ funkcija dostiže maksimum.

3278. U tački $(0,\ 0)$ nema ekstremuma; u tački $(1,\ 1)$ funkcija dostiže minimum.

3279. Najveću i najmanju vrednost funkcija dostiže na granici oblasti: najveću $z=4$ u tačkama $(2,\ 0)$ i $(-2,\ 0$, a najmanju, $z=-4$, u tačkama $(0,\ 2)$ i $(0,\ -2)$. U stacionarnoj tački $(0,\ 0)$ nema ekstremuma.

3280. Najveća vrednost $z=17$ u tački $(1,\ 2)$; najmanja vrednost $z=-3$ u tački $(1,\ 0)$; stacionarna tačka $(-4,\ 6)$ leži van date oblasti.

3281. Najveća vrednost $z=4$ u stacionarnoj tački $(2,\ 1)$ (ova tačka je, prema tome, tačka ekstremuma); najmanja vrednost $z=-64$ u tački $(4,\ 2)$ koja leži na granici oblasti.

3282. Najmanju vrednost $z=0$ funkcija dostiže u tački $(0,\ 0)$; najveću vrednost $z=\frac{3}{e}$ u tačkama $(0,\ \pm 1)$.

3283. $z_{\max}=\frac{3}{2}\sqrt{3}$ u tački $\left(\frac{\pi}{3},\ \frac{\pi}{3}\right)$, $z_{\min}=0$ u tački $(0,\ 0)$ (na granici oblasti).

3284. Svi sabirci moraju biti jednaki među sobom.

3285. Svi množitelji moraju biti jednaki među sobom.

3286. $\left(\frac{8}{5},\ \frac{16}{5}\right)$. **3287.** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$.

3288. $x=\frac{\sum_{i=1}^{n}x_i}{n}$, $y=\frac{\sum_{i=1}^{n}y_i}{n}$. **3289.** $(3,\ \sqrt{39},\ 0)$; $(3,\ -\sqrt{39},\ 0)$.

3290. Kocka. **3291.** U tački $(1,\ 1)$ je $z=2$ — minimum.

3292. $(a,\ a)$ ili $(-a,\ -a)$, $z=a^2$ (maksimum), $(a,\ -a)$ ili $(-a,\ a)$, $z=-a^2$ (minimum).

3293. $(-a\sqrt{2},\ -a\sqrt{2})$, $z=-\frac{\sqrt{2}}{a}$ (minimum), $(a\sqrt{2},\ a\sqrt{2})$, $z=\frac{\sqrt{2}}{a}$ (maksimum).

3294. Stacionarne tačke $x=-\frac{1}{2}\operatorname{Arctg}\frac{b}{a}$, $y=\frac{1}{2}\operatorname{Arctg}\frac{a}{b}$.

3295. $(3, 3, 3)$, $u=9$ (minimum).

3296. Kad su vrednosti dveju nezavisno promenljivih $=2$, a vrednost treće $=1$, funkcija dostiže minimum $=4$; kad su vrednosti dveju nezavisno promenljivih $=\frac{4}{3}$, a vrednost treće $\frac{7}{3}$, funkcija dostiže maksimum $=\frac{112}{27}$.

3297*. Treba naći minimum funkcije $\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}$ pod uslovom $x_1+x_2+\cdots+x_n=A$. Uopšte, važi relacija $\frac{\Sigma x_i^k}{n}>\left(\frac{\Sigma x_i}{n}\right)^k$ za $k>1$.

3299. $u_{\min}=\frac{abc}{bc+ca+ab}$ za $x=\frac{bc}{bc+ca+ab}$; $y=\frac{ac}{bc+ac+ab}$; $z=\frac{ab}{bc+ac+ab}$.

3300. $x=\pm\frac{1}{2}$, $y=\pm\frac{1}{3}$, $z=\pm\frac{1}{6}$. 3301. $\left(\frac{21}{13}, 2, \frac{63}{26}\right)$.

3302. $(3, -1, 1)$. 3303. a) $(-2, 0, 0)$; b) $(2, 0, 0)$.

3304. Kocka. 3305. Kocka. 3306. $\frac{8abc}{3\sqrt{3}}$.

3307. Ako je R poluprečnik osnove šatora, H — visina cilindričnog dela, a h — visina konusnog završetka, onda moraju važiti sledeće relacije: $R=\frac{h\sqrt{5}}{2}$, $H=\frac{h}{2}$.

3308. Ako je b osnovica, l — krak, a α — ugao na osnoviii trapeza, onda mora biti $l=b=\frac{2\sqrt{A}}{\sqrt[4]{3}}$, $\alpha=\frac{\pi}{3}$, pri čemu je A data površina preseka; tada je kvašena površina $u=2\sqrt[4]{3}\cdot\sqrt{A}\approx 2{,}632\sqrt{A}$.

3309. Kocka. 3310. Svaka od osnovnih ivica je $2\alpha+\sqrt[3]{2v}$, a visina je dva puta manja $\alpha+\frac{1}{2}\sqrt[3]{2v}$.

3311. a^3 (kocka). 3312. Minimalna površina ima vrednost $3\sqrt{3}\,ab$.

3313. $x=\pm\frac{4}{\sqrt{5}}$, $y=\pm\frac{3}{\sqrt{5}}$. 3314. $\left(-\frac{5}{9}, -\frac{1}{9}\right)$. 3315. $(3, 5)$. 3316. $z_{\max}=2$.

3317. Stranice trougla su $\sqrt{2S}$, $\sqrt{2S}$ i $2\sqrt{S}$.

3318. Visina je $\frac{H}{3}$, osnovne ivice su $\frac{2a\sqrt{2}}{3}$ i $\frac{2b\sqrt{2}}{3}$, a zapremina $V=\frac{8}{27}abH$.

3319. Tetraedar.

3320. Normala elipse u traženoj tački mora biti normalna na pravoj koja spaja date tačke.

3321. Normalu povući u tački sa koordinatama

$$\left(\pm a\sqrt{\frac{a}{a+b}},\ \pm b\sqrt{\frac{b}{a+b}}\right).$$

3322. $\left(9, \frac{1}{8}, \frac{3}{8}\right)$; $\left(-9, -\frac{1}{8}, -\frac{3}{8}\right)$. 3323. $2\sqrt{2}$.

3439. 1) $\{-2, 1\}$; 2) $\{10xy-3y^3,\ 5x^2-9xy^2+4y^3\}$.

3440. 1) $6i+4j$; 2) $\frac{1}{3}(2i+j)$; 3) $\frac{-y_0 i+x_0 j}{x_0^2+y_0^2}$.

3441. 1) $\operatorname{tg}\varphi\approx 0{,}342$, $\varphi\approx 18°52'$; 2) $\operatorname{tg}\varphi\approx 4{,}87$, $\varphi\approx 78°24'$.

3442. Negativni deo y-ose.

3443. 1) $\cos\alpha\approx 0{,}99$, $\alpha\approx 8°$; 2) $\cos\alpha\approx -0{,}199$, $\alpha\approx 101°30'$.

3444. 1) $\left(-\frac{1}{3}, \frac{3}{4}\right)$; $\left(\frac{7}{3}, -\frac{3}{4}\right)$; 2) tačke koje leže na krugu $x^2+y^2=\frac{2}{3}$.

3447. 1) $\{3x_0^2y_0^2z_0,\ 2x_0^3y_0z_0,\ x_0^3y_0^2\}$; 2) $\frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}=\frac{r}{|r|}$, gde je r — vektor položaja tačke.

3450. 1) $2r$; 2) $2\frac{r}{|r|}$; 3) $2F'(r^2)r$; 4) $a(br)+b(ar)$; 5) $a\times b$.

3451. 1) 0; 2) $\frac{\sqrt{2}}{2}$; 3) $\sqrt{5}$; 4) $\frac{\cos\alpha+\sin\alpha}{2}$.

3452. $\frac{\sqrt{2}}{3}$. 3453. $\frac{1}{2}$. 3455. 1) 5; 2) $\frac{98}{13}$. 3456. 22. 3459. $\frac{1}{r^2}$.

Ekstremne vrijednosti f-ja dviju promjenjivih

Neka je data f-ja $z=f(x,y)$.

$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=0\\ \frac{\partial z}{\partial y}&=0\end{aligned}\right\}$$

rješenjem sistema dobijemo stacionarne tačke koje mogu ali i ne moraju biti ekstrem

SISTEM

npr. $M(p_1,p_2)$ je jedna stacionarna tačka.

$A=\dfrac{\partial^2 z(p_1,p_2)}{\partial x^2}$

$B=\dfrac{\partial^2 z(p_1,p_2)}{\partial x\partial y}$

$C=\dfrac{\partial^2 z(p_1,p_2)}{\partial y^2}$

$D=AC-B^2$

$D>0$ f-ja ima ekstrem u tački $M(p_1,p_2)$

a) $A>0$ imamo z_{min}

b) $A<0$ imamo z_{max}

$D<0$ f-ja nema ekstrem

$D=0$ potrebno ispitati ponašanje f-je u okolini stacionarne tačke:

$\Delta z(M)=z(p_1+\varepsilon,p_2+\omega)-z(p_1,p_2)$ – priraštaj f-je

$\Delta z\geq 0\ \ \forall\varepsilon;\forall\omega \Rightarrow$ u tački M f-ja ima minimum

$\Delta z\leq 0\ \ \forall\varepsilon;\forall\omega \Rightarrow$ u tački M f-ja ima maksimum

\# Naći ekstreme f-je $z=x^2-2x-y-\ln(2-y)+4$.

R: $\mathcal{D}: 2-y>0$

$\dfrac{\partial z}{\partial x}=2x-2$

$\dfrac{\partial z}{\partial y}=-1-\dfrac{1}{2-y}\cdot(-1)=\dfrac{1}{2-y}-1$

$2x-2=0$

$\dfrac{1}{2-y}-1=0$

$x=1,\ y=1$

Tačka $M(1,1)$ je stacionarna tačka (kandidat za ekstrem)

$(2-y)^{-1}$

$\dfrac{\partial^2 z}{\partial x^2}=2$

$\dfrac{\partial^2 z}{\partial x\partial y}=0$

$\dfrac{\partial^2 z}{\partial y^2}=(-1)(2-y)^{-2}\cdot(-1)=\dfrac{1}{(2-y)^2}$

$M(1,1)$

$A=2,\ B=0,\ C=1$

$D=AC-B^2=2>0$

F-ja ima ekstrem.

$A>0\Rightarrow$ f-ja ima minimum

$z_{min}(1,1)=1-2-1-\ln 1+4=-2+4=2$

Naći ekstreme f-je $z = x^3 - 5xy + 5y^2 + 7x - 15y$.

Rj. Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 3x^2 - 5y + 7 \qquad \frac{\partial z}{\partial y} = 10y - 5x - 15$$

$$\begin{aligned} 3x^2 - 5y + 7 &= 0 \quad /\cdot 2 \\ -5x + 10y - 15 &= 0 \\ \hline 6x^2 - 10y + 14 &= 0 \\ -5x + 10y - 15 &= 0 \quad + \\ \hline \end{aligned}$$

$$6x^2 - 5x - 1 = 0$$

$$D = 25 + 24 = 49$$

$$x_{1,2} = \frac{5 \pm 7}{2\cdot 6}$$

$$x_1 = \frac{-2}{2\cdot 6} = \frac{-1}{6}, \quad x_2 = \frac{12}{12} = 1$$

$$6\left(x + \frac{1}{6}\right)(x - 1) = 0$$

Za $x_1 = -\frac{1}{6} \Rightarrow -5\cdot\left(-\frac{1}{6}\right) + 10y - 15 = 0$

$$10y = 15 - \frac{5}{6}$$

$$10y = \frac{90-5}{6} = \frac{85}{6}$$

$$y = \frac{\cancel{85}^{17}}{\cancel{10}_2\, 6} = \frac{17}{12}$$

$x_2 = 1 \Rightarrow -5 + 10y - 15 = 0$

$$10y = 20$$

$$y = 2$$

Stacionarne tačke su $(1,2)$ i $\left(-\frac{1}{6}, \frac{17}{12}\right)$

$$\frac{\partial^2 z}{\partial x^2} = 6x \qquad \frac{\partial^2 z}{\partial x \partial y} = -5 \qquad \frac{\partial^2 z}{\partial y^2} = 10$$

Za $M_1(1,2)$

$A = 6,\ B = -5,\ C = 10,\ D = AC - B^2 = 60 - 25 > 0$

f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$$z_{min}(1,2) = 1 - 10 + 20 + 7 - 30 = 8 + 10 - 30 = 8 - 20 = -12$$

Za $M_2\left(-\frac{1}{6}, \frac{17}{12}\right)$

$A = -1,\ B = -5,\ C = 10,\ D = AC - B^2 = -10 - 25 = -35$

F-ja u ovoj tački nema ekstrem

Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2 + y^2 + 1)$.

Rj. Izračunajmo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{\cancel{2}}\cdot\frac{1}{x^2+y^2+1}\cdot\cancel{2}x = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{\cancel{2}}\cdot\frac{1}{x^2+y^2+1}\cdot\cancel{2}y = 1 - \frac{3y}{x^2+y^2+1}$$

Nađimo stacionarne tačke

$$\begin{aligned} 1 - \frac{3x}{x^2+y^2+1} &= 0 \\ 1 - \frac{3y}{x^2+y^2+1} &= 0 \\ \hline 3x = 3y &\Rightarrow x = y \end{aligned}$$

$$1 - \frac{3x}{2x^2+1} = 0 \quad /\cdot 2x^2+1$$

$$2x^2 + 1 - 3x = 0$$

$$2x^2 - 3x + 1 = 0$$

$$D = 9 - 8 = 1$$

$$x_{1,2} = \frac{3 \pm 1}{4}$$

$$x_1 = \frac{2}{4} = \frac{1}{2}$$

$$x_2 = 1$$

Stacionarne tačke su $M_1\left(\frac{1}{2}, \frac{1}{2}\right)$ i $M_2(1,1)$

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = 0 - 3\cdot\frac{1\cdot(x^2+y^2+1) - x\cdot 2x}{(x^2+y^2+1)^2} = -3\cdot\frac{-x^2+y^2+1}{(x^2+y^2+1)^2} = 3\,\frac{x^2-y^2-1}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0 - 3x\cdot(-1)(x^2+y^2+1)^{-2}\cdot 2y = 6\,\frac{xy}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = 0 - 3\,\frac{1\cdot(x^2+y^2+1) - y\cdot 2y}{(x^2+y^2+1)^2} = -3\,\frac{x^2-y^2+1}{(x^2+y^2+1)^2}$$

Za $M_1\left(\frac{1}{2}, \frac{1}{2}\right)$, $A = 3\cdot\frac{-1}{\left(\frac{1}{2}+1\right)^2} = \frac{-3}{\frac{9}{4}} = \frac{-12}{9} = -\frac{4}{3}$, $B = \frac{2}{3}$, $C = -\frac{4}{3}$

$D = AC - B^2 = \frac{16}{9} - \frac{4}{9} > 0$ f-ja ima ekstrem u tački M_1

$A < 0$ f-ja ima minimum $z_{min}\left(\frac{1}{2}, \frac{1}{2}\right) = 1 - \frac{3}{2}\ln\frac{3}{2}$

Za $M_2(1,1)$, $A = -\frac{1}{3}$, $B = \frac{2}{3}$, $C = -\frac{1}{3}$

$D = AC - B^2 = \frac{1}{9} - \frac{4}{9} < 0$ f-ja u tački M_2 nema ekstrem

Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2+y^2+1)$.

Rj: Pronađimo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2}\cdot\frac{2x}{x^2+y^2+1} = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2}\cdot\frac{2y}{x^2+y^2+1} = 1 - \frac{3y}{x^2+y^2+1}$$

Pronađimo stacionarne tačke

$$\left.\begin{array}{l} \frac{\partial z}{\partial x} = 0 \Rightarrow 1 = \frac{3x}{x^2+y^2+1} \\ \frac{\partial z}{\partial y} = 0 \Rightarrow 1 = \frac{3y}{x^2+y^2+1} \end{array}\right\} \Rightarrow x = y \text{ (deljenjem jednačina)}$$

Sad imamo $x = y$ i $1 = \frac{3x}{x^2+y^2+1} \Rightarrow 1 = \frac{3x}{2x^2+1} \Rightarrow 2x^2-3x+1=0$

$D = 9-8 = 1$

$x_1 = 1,\ x_2 = \frac{1}{2}$

Stacionarne tačke su $M_1(1,1)$ i $M_2(\frac{1}{2}, \frac{1}{2})$.

Pronađimo druge parcijalne izvode.

$$\frac{\partial^2 z}{\partial x^2} = \left(1 - \frac{3x}{x^2+y^2+1}\right)'_x = \frac{-3(x^2+y^2+1)+3x\cdot 2x}{(x^2+y^2+1)^2} = \frac{3x^2-3y^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(1 - \frac{3y}{x^2+y^2+1}\right)'_y = \left|\begin{array}{l}\text{zbog}\\ \text{simetričnosti}\end{array}\right| = \frac{3y^2-3x^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{3x\cdot 2y}{(x^2+y^2+1)^2} = \frac{6xy}{(x^2+y^2+1)^2}$$

Za tačku $M_1(1,1)$: $A = -\frac{3}{9} = -\frac{1}{3}$, $B = \frac{6}{9} = \frac{2}{3}$, $C = -\frac{3}{9} = -\frac{1}{3}$, $D = AC - B^2$

$D = \frac{1}{9} - \frac{4}{9} < 0 \Rightarrow$ u M_1 f-ja nema ekstrema

Za tačku $M_2(\frac{1}{2}, \frac{1}{2})$: $A = \frac{-3}{(\frac{3}{2})^2} = -\frac{3}{\frac{9}{4}} = -\frac{12}{9} = -\frac{4}{3} \Rightarrow C = -\frac{4}{3}$

$B = \frac{\frac{3}{2}}{\frac{9}{4}} = \frac{12}{18} = \frac{2}{3}$, $D = AC - B^2 = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} = \frac{4}{3} > 0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A < 0 \Rightarrow$ u M_2 f-ja ima maximum. $z_{max}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} + \frac{1}{2} - 3\ln(\frac{1}{4} + \frac{1}{4} + 1) = 1 - \ln\frac{27}{8}$

Naći ekstreme f-je $z = \frac{8}{x} + \frac{x^2}{y} + y + 1$.

Rj. Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 8\cdot(-1)x^{-2} + 2\frac{x}{y} = \frac{-8}{x^2} + 2\frac{x}{y}$$

$$\frac{\partial z}{\partial y} = x^2\cdot(-1)y^{-2} + 1 = \frac{-x^2}{y^2} + 1$$

$$-\frac{8}{x^2} + \frac{2x}{y} = 0$$

$$-\frac{x^2}{y^2} + 1 = 0$$

$$\frac{8}{x^2} - 2\frac{x}{y} = 0$$

$$\frac{x^2}{y^2} = 1 \Rightarrow \left(\frac{x}{y}\right)^2 = 1$$

Prema tome $\frac{x}{y} = 1$ i $\frac{x}{y} = -1$

Za $\frac{x}{y} = 1 \Rightarrow \frac{8}{x^2} - 2\cdot 1 = 0$

$\frac{8}{x^2} = 2 \quad /\cdot x^2\ (x \neq 0)$

$2x^2 = 8$

$x^2 = 4$

$x_1 = -2,\ x_2 = 2$

$x_1 = -2 \Rightarrow \frac{x}{y} = 1$

$y = -2$

$(-2,-2)$

za $x_2 = 2 \Rightarrow$

$\Rightarrow \frac{x}{y} = 1$

$y_2 = 2$

$(2,2)$

Za $\frac{x}{y} = -1$ imamo

$\frac{8}{x^2} + 2 = 0$

$\frac{8}{x^2} = -2 \quad /\cdot x^2\ (x \neq 0)$

$-2x^2 = 8$

ova jednačina nema rešenja u skupu realnih brojeva

Stacionarne tačke su $M_1(-2,-2)$ i $M_2(2,2)$.

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = (-8)(-2)x^{-3} + \frac{2}{y} = \frac{16}{x^3} + \frac{2}{y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\cdot(-1)y^{-2} = \frac{-2x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2\cdot(-2)y^{-3} = \frac{2x^2}{y^3}$$

Za $M_2(2,2)$

$A = 2+1 = 3$, $B = \frac{-4}{4} = -1$, $C = \frac{8}{8} = 1$

$D = AC - B^2 = 3 - 1 = 2 > 0$ f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2) = 4 + 2 + 2 + 1 = 9$

Za $M_1(-2,-2)$

$A = \frac{16}{-8} + \frac{2}{-2} = -2 - 1 = -3$

$B = \frac{-2\cdot(-2)}{4} = 1$, $C = \frac{2\cdot 4}{-8} = -1$

$D = AC - B^2 = 3 - 1 = 2 > 0$.

f-ja u tački $M_1(-2,-2)$ ima ekstrem,

$A < 0$ f-ja ima maksimum

$z_{max}(-2,-2) = -4 - 2 - 2 + 1 = -7$

Naći ekstreme f-je $z=(x^2+y)\sqrt{e^y}$.

Rj. $\frac{\partial z}{\partial x} = 2x\sqrt{e^y}$

$$\frac{\partial z}{\partial y} = \sqrt{e^y} + (x^2+y)\frac{1}{2\sqrt{e^y}}\cdot e^y = \sqrt{e^y} + (x^2+y)\cdot\frac{1}{2}\sqrt{e^y}$$

$$= \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right)\sqrt{e^y} = \frac{1}{2}(x^2+y+2)\sqrt{e^y}$$

$$\begin{aligned} 2x\sqrt{e^y} &= 0 \\ \frac{1}{2}(x^2+y^2+1)\sqrt{e^y} &= 0 \end{aligned}$$

$e^y > 0 \ \forall y \in \mathbb{R}$

prema tome $x=0$

$\sqrt{e^y} > 0 \ \forall x \in \mathbb{R}$

$x^2+y+2=0$

$x=0 \Rightarrow y+2=0$

$y=-2$

$M(0,-2)$ je stacionarna tačka (kandidat za ekstrem)

$$\frac{\partial^2 z}{\partial x^2} = 2\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\frac{1}{2\sqrt{e^y}}\cdot e^y = x\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\sqrt{e^y} + \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right)\frac{e^y\cdot\sqrt{e^y}}{2\sqrt{e^y}\cdot\sqrt{e^y}} = \frac{1}{2}\sqrt{e^y}\left(\frac{1}{2}x^2 + \frac{1}{2}y + 2\right)$$

$M(0,-2)$

$A = 2\sqrt{e^2} = 2\frac{1}{\sqrt{e^2}}$

$B = 0$

$C = \frac{1}{2}\sqrt{e^{-2}}\left(\frac{1}{2}0 + \underbrace{\frac{1}{2}(-2)}_{-1} + 2\right) = \frac{1}{2}\sqrt{\frac{1}{e^2}}$

$D = AC - B^2 = \frac{2}{\sqrt{e^2}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{e^2}} = \frac{1}{e^2}$

$D>0 \Rightarrow$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(0,-2) = (0-2)\sqrt{e^{-2}} = (-2)\cdot\frac{1}{\sqrt{e^2}} \approx -0.7358$$

Naći ekstreme f-je $z=e^{-2x^2}(x-y^2)$.

Rj. Nađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = e^{-2x^2}\cdot(-4)x(x-y^2) + e^{-2x^2}\cdot 1 = e^{-2x^2}(-4x^2+4xy^2+1)$$

$$\frac{\partial z}{\partial y} = e^{-2x^2}\cdot(-2)y = -2ye^{-2x^2}$$

$$\begin{aligned} e^{-2x^2}(-4x^2+4xy^2+1) &= 0 \\ -2y\,e^{-2x^2} &= 0 \end{aligned}$$

e^{-2x^2} je uvijek pozitivno

$$\begin{aligned} -4x^2+4xy^2+1 &= 0 \\ -2y &= 0 \end{aligned} \Rightarrow y=0$$

$-4x^2+1=0$

$x^2 = \frac{1}{4} \Rightarrow x_1 = -\frac{1}{2}, \ x_2 = \frac{1}{2}$

Stacionarne tačke su $M_1(-\frac{1}{2},0)$ i $M_2(\frac{1}{2},0)$

$$\frac{\partial^2 z}{\partial x^2} = e^{-2x^2}\cdot(-4x)(-4x^2+4xy^2+1) + e^{-2x^2}(-8x+4y^2) =$$

$$= e^{-2x^2}(16x^3-16x^2y^2-4x-8x+4y^2) = e^{-2x^2}(16x^3-16x^2y^2-12x+4y^2)$$

$$= 4e^{-2x^2}(4x^3-4x^2y^2-3x+y^2)$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{-2x^2}(8xy) = 8xye^{-2x^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -2e^{-2x^2}$$

Za tačku $M_1(-\frac{1}{2},0)$

$A = 4e^{-2\cdot\frac{1}{4}}\left(4\cdot\left(-\frac{1}{8}\right) - 4\cdot\frac{1}{4}\cdot 0 - 3\cdot\left(-\frac{1}{2}\right) + 0\right) = 4e^{-\frac{1}{2}}\left(-\frac{1}{2}+\frac{3}{2}\right) = \frac{4}{\sqrt{e}}$

$B=0$, $C = -2e^{-2\cdot\frac{1}{4}} = -2e^{-\frac{1}{2}} = \frac{-2}{\sqrt{e}}$

$D = AC - B^2 = \frac{-8}{e} < 0$

f-ja z u tački M_1 nema ekstrem

Za tačku $M_2(\frac{1}{2},0)$

$A = 4e^{-2\cdot\frac{1}{4}}\left(4\cdot\frac{1}{8} - 0 - 3\cdot\frac{1}{2} + 0\right) =$

$= 4e^{-\frac{1}{2}}\left(\frac{1}{2} - \frac{3}{2}\right) = \frac{-4}{\sqrt{e}}$

$B=0$, $C = -2e^{-2\cdot\frac{1}{4}} = \frac{-2}{\sqrt{e}}$

$D = AC - B^2 = \frac{8}{e} > 0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A<0 \Rightarrow z_{max}(\frac{1}{2},0) = e^{-2\cdot\frac{1}{4}}\left(\frac{1}{2} - 0\right) = \frac{1}{2}\cdot e^{-\frac{1}{2}} = \frac{1}{2\sqrt{e}}$

Odrediti ekstremne vrijednosti f-je

$$z = \frac{xy}{2} + (47-x-y)\left(\frac{x}{3}+\frac{y}{4}\right).$$

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{2}y + (-1)\left(\frac{x}{3}+\frac{y}{4}\right) + (47-x-y)\cdot\frac{1}{3} = \frac{1}{2}y - \frac{1}{3}x - \frac{1}{4}y + \frac{47}{3} - \frac{1}{3}x - \frac{1}{3}y$$

$$= -\frac{2}{3}x + \frac{6-3-4}{12}y + \frac{47}{3} = -\frac{2}{3}x - \frac{1}{12}y + \frac{47}{3}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}x + (-1)\left(\frac{x}{3}+\frac{y}{4}\right) + (47-x-y)\cdot\frac{1}{4} = \frac{1}{2}x - \frac{1}{3}x - \frac{1}{4}y + \frac{47}{4} - \frac{1}{4}x - \frac{1}{4}y$$

$$= -\frac{1}{12}x - \frac{1}{2}y + \frac{47}{4}$$

$$-\frac{2}{3}x - \frac{1}{12}y + \frac{47}{3} = 0 \quad /\cdot 12$$
$$-\frac{1}{12}x - \frac{1}{2}y + \frac{47}{4} = 0 \quad /\cdot 12$$

$$-8x - y + 188 = 0$$
$$-x - 6y + 141 = 0$$

$$-8x - y + 188 = 0$$
$$x = -6y + 141$$

$$-8(-6y+141) - y + 188 = 0$$
$$48y - 1128 - y + 188 = 0$$
$$47y = 940$$
$$y = 20$$
$$x = -6y + 141 = -120 + 141 = 21$$

Stacionarna tačka je $M(21,20)$.

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{3} \qquad \frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{12} \qquad \frac{\partial^2 z}{\partial y^2} = -\frac{1}{2}$$

$D = AC - B^2$

$M(21,20)$

$A = -\frac{2}{3}, \quad B = -\frac{1}{12}, \quad C = -\frac{1}{2}$

$D = \frac{2}{6} - \frac{1}{144} = \frac{1}{3} - \frac{1}{144} > 0$

f-ja z ima ekstrem

$A < 0$ f-ja ima maksimum

$$z_{max}(21,20) = 21\cdot 10 + (47-41)(7+5) = 210 + 6\cdot 12 = 210 + 72 = 282$$

$z_{max}(21,20) = 282$ traženi ekstrem f-je

Nađi ekstreme f-je $z = x^4 + y^4 - 2x^2$.

Rj.

$$\frac{\partial z}{\partial x} = 4x^3 - 4x \qquad \frac{\partial z}{\partial y} = 4y^3$$

$$4x^3 - 4x = 0 \quad /:4$$
$$4y^3 = 0 \quad /:4$$
$$x^3 - x = 0$$
$$y^3 = 0$$

$$x(x^2-1) = 0$$
$$y^3 = 0$$
$$x(x-1)(x+1) = 0$$
$$y^3 = 0$$

$y = 0 \ \wedge \ (x_1 = 0, x_2 = 1, x_3 = -1)$

Stacionarne tačke f-je su $M_1(-1,0)$, $M_2(0,0)$ i $M_3(1,0)$.

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 4 \qquad \frac{\partial^2 z}{\partial x \partial y} = 0 \qquad \frac{\partial^2 z}{\partial y^2} = 12y^2$$

za $M_1(-1,0)$, $A = 8$, $B = 0$, $C = 0$

$D = 0$ ispitujemo ponašanje f-je u okolini tačke $M_1(-1,0)$

$$\Delta z = z(-1+\varepsilon, 0+\omega) - z(-1,0) =$$
$$= (-1+\varepsilon)^4 + \omega^4 - 2(-1+\varepsilon)^2 - \left[(-1)^4 + 0^4 - 2(-1)^2\right]$$
$$= 1 - 4\varepsilon + 6\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4 + \omega^4 - 2(1 - 2\varepsilon + \varepsilon^2) - (1-2)$$
$$= 1 - 4\varepsilon + 6\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4 + \omega^4 - 2 + 4\varepsilon - 2\varepsilon^2 + 1$$
$$= \varepsilon^4 - 4\varepsilon^3 + 4\varepsilon^2 + \omega^4 = \varepsilon^2(\varepsilon^2 - 4\varepsilon + 4) + \omega^4$$
$$= \varepsilon^2(\varepsilon - 2)^2 + \omega^4 \geq 0 \quad \text{za } \forall\varepsilon \text{ i } \forall\omega$$

paskalov trougao

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

f-ja ima minimum u tački $M_1(-1,0)$, $z_{min} = -1$

za $M_2(0,0)$, $A = -4$, $B = 0$, $C = 0$, $D = AC - B^2 = 0$

ispitujemo ponašanje f-je u okolini tačke

$$\Delta z = z(0+\varepsilon, 0+\omega) - z(0,0) = \varepsilon^4 + \omega^4 - 2\varepsilon^2 = \varepsilon^2(\varepsilon^2 - 2) + \omega^4$$

$\varepsilon = 0: \quad \Delta z = \omega^4$

$\omega = 0: \quad \Delta z = \varepsilon^2(\varepsilon^2 - 2) \Rightarrow \Delta z < 0$ za $\varepsilon^2 < 2$; $\Delta z > 0$ za $\varepsilon^2 > 2$

Priraštaj f-je je promjenjivog znaka pa f-ja nema ekstrem u tački M_2

za $M_3(1,0)$, $A = 8$, $B = 0$, $C = 0$, $D = AC - B^2 = 0$ ispitujemo ponašanje f-je u okolini tačke

$$\Delta z = z(1+\varepsilon, 0+\omega) - z(1,0) = (1+\varepsilon)^4 + \omega^4 - 2(1+\varepsilon)^2 - (1-2)$$
$$= 1 + 4\varepsilon + 6\varepsilon^2 + 4\varepsilon^3 + \varepsilon^4 + \omega^4 - 2 - 4\varepsilon - 2\varepsilon^2 + 1 = \varepsilon^4 + 4\varepsilon^3 + 4\varepsilon^2 + \omega^4$$

$$\Delta z = \varepsilon^2(\varepsilon + 2)^2 + \omega^4 \geq 0 \quad \forall\varepsilon \text{ i } \forall\omega$$

f-ja z u tački M_3 ima min

$z_{min} = -1$

Naći ekstreme f-je $z=(2x^2+3y^2)e^{-(x^2+y^2)}$.

Rj.

$$\frac{\partial z}{\partial x} = 4x\cdot e^{-x^2-y^2} + (2x^2+3y^2)e^{-x^2-y^2}\cdot(-2x) = (4x-4x^3-6xy^2)e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y} = 6y e^{-x^2-y^2} + (2x^2+3y^2)e^{-x^2-y^2}\cdot(-2y) = (6y-4x^2y-6y^3)e^{-x^2-y^2}$$

$$2x(2-2x^2-3y^2)e^{-x^2-y^2}=0$$
$$2y(3-2x^2-3y^2)e^{-x^2-y^2}=0$$

$e^{-x^2-y^2}\neq 0 \quad \forall(x,y\in R)$

$x=0$; $y=0$, $M_1(0,0)$

ili

$x=0$ i $3-2x^2-3y^2=0$

$3y^2=3$, $y^2=1$, $y_{1,2}=\pm 1$

$M_2(0,-1)$, $M_3(0,1)$

ili

$2-2x^2-3y^2=0$; $y=0$

$2x^2=2$, $x^2=1$, $x_{1,2}=\pm 1$

$M_4(-1,0)$, $M_5(1,0)$

ili

$$\begin{array}{r} 2-2x^2-3y^2=0 \\ -\ \ 3-2x^2-3y^2=0 \\ \hline -1=0 \end{array}$$

sistem nema rješenja

Stacionarne tačke su M_1, M_2, M_3, M_4 i M_5.

$$\frac{\partial^2 z}{\partial x^2} = (4-12x^2-6y^2)e^{-x^2-y^2} + (4x-4x^3-6xy^2)e^{-x^2-y^2}(-2x) = (8x^4+12x^2y^2-20x^2-6y^2+4)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = (-12xy)e^{-x^2-y^2} + (4x-4x^3-6xy^2)e^{-x^2-y^2}(-2y) = (-20xy+8x^3y+12xy^3)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = (6-4x^2-18y^2)e^{-x^2-y^2} + (6y-4x^2y-6y^3)e^{-x^2-y^2}\cdot(-2y) = (-30y^2+12y^4+8x^2y^2-4x^2+6)e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24>0$ ima ekstrem

$A>0$ ima minimum, $z_{min}(0,0)=0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum, $z_{max}(0,-1)=3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum $z_{max}(0,1)=3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2}<0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$,

f-ja u tački $M_5(1,0)$ nema ekstrem

Naći stacionarne tačke f-je $z=xy\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = y\ln(x^2+y^2) + xy\cdot\frac{1}{x^2+y^2}\cdot 2x = y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = x\ln(x^2+y^2) + xy\cdot\frac{1}{x^2+y^2}\cdot 2y = x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2}$$

$$y\ln(x^2+y^2) + \frac{2x^2y}{x^2+y^2}=0$$
$$x\ln(x^2+y^2) + \frac{2xy^2}{x^2+y^2}=0$$

$$y\left(\ln(x^2+y^2) + \frac{2x^2}{x^2+y^2}\right)=0$$
$$x\left(\ln(x^2+y^2) + \frac{2y^2}{x^2+y^2}\right)=0$$

$y=0$ ili $\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}=0$

$x=0$ ili $\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0$

$y=0$; $x=0$ $M_1(0,0)$ za M_1 f-ja nije definisana

ili

$y=0$; $\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0$

$\ln x^2=0$, $x^2=1$, $x_1=-1$, $x_2=1$

$M_2(-1,0)$, $M_3(1,0)$

ili

$\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}=0$ i $x=0$

$\ln y^2=0$, $y_1=-1$ $y_2=1$

$M_4(0,-1)$, $M_5(0,1)$

ili

$$\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}=0 \quad (1)$$
$$\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0 \quad (2)$$

$(1)-(2)$: $\frac{2x^2}{x^2+y^2}-\frac{2y^2}{x^2+y^2}=0$

$2x^2-2y^2=0$

$2(x^2-y^2)=0$

$2(x-y)(x+y)=0$

$x=y$ ili $x=-y$.

za $y=-x$: $\ln(2x^2)+1=0$

$x_{1,2}=\pm\frac{1}{\sqrt{2e}}$

$M_8\left(-\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$, $M_9\left(\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$

za $x=y$: $\ln(2x^2)+1=0$

$\ln(2x^2)=-1$

$e^{-1}=2x^2$, $x^2=\frac{1}{2e}$, $x_{1,2}=\pm\frac{1}{\sqrt{2e}}$

$M_6\left(-\frac{1}{\sqrt{2e}},-\frac{1}{\sqrt{2e}}\right)$, $M_7\left(\frac{1}{\sqrt{2e}},\frac{1}{\sqrt{2e}}\right)$

Stacionarne tačke su: $M_2, M_3, M_4, M_5, M_6, M_7, M_8$ i M_9.

Ekstremi f-ja tri promjenjive

Neka je data f-ja $u=f(x,y,z)$

$$\left.\begin{array}{l} u'_x=0 \\ u'_y=0 \\ u'_z=0 \end{array}\right\}$$ rješenjem sistema dobijemo stacionarne tačke koje mogu ali i ne moraju biti ekstrem

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npr. $M(p_1,p_2,p_3)$ je jedna stacionarna tačka

I način: Silvesterov kriterij

$a_{11}=u''_{xx}(M)$ $\quad a_{21}=u''_{yx}(M)$ $\quad a_{31}=u''_{zx}(M)$

$a_{12}=u''_{xy}(M)$ $\quad a_{22}=u''_{yy}(M)$ $\quad a_{32}=u''_{zy}(M)$

$a_{13}=u''_{xz}(M)$ $\quad a_{23}=u''_{yz}(M)$ $\quad a_{33}=u''_{zz}(M)$

$$T=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

a) $a_{11}>0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0$, $\det(T)>0 \Rightarrow M$ je tačka minimuma

b) $a_{11}<0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}>0$, $\det(T)<0 \Rightarrow M$ je tačka maksimuma

c) $a_{11}=0$ ili $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}=0$ ili $\det T=0 \Rightarrow$ potrebno je ispitati f-ju u okolini tačke M (formira se priraštaj Δu, $\Delta u=u(p_1+\varepsilon, p_2+\omega, p_3+\delta)-u(p_1,p_2,p_3)$)

d) za sve ostale kombinacije znakova u tački M nema ekstrema.

II način: Pomoću diferencijala d^2u

a) Ako je $d^2u(M)>0$ tada u tački M f-ja ima min

b) Ako je $d^2u(M)<0$ tada u tački M f-ja ima max

c) $d^2u(M)$ promjenjivog znaka $\Rightarrow$ u tački M f-ja nema ekstrema

\# Nađi ekstreme f-je $u=\sin x+\sin y+\sin z-\sin(x+y+z)$ gdje su $0\le x\le\pi$, $0\le y\le\pi$ i $0\le z\le\pi$.

Rj.

$u'_x=\cos x-\cos(x+y+z)$

$u'_y=\cos y-\cos(x+y+z)$

$u'_z=\cos z-\cos(x+y+z)$

$\cos x-\cos(x+y+z)=0$

$\cos y-\cos(x+y+z)=0$

$\cos z-\cos(x+y+z)=0$

$\cos x=\cos y=\cos z=\cos(x+y+z)$

Možemo li imati dva različita ugla koja nisu u istom kvadrantu (prvom ili drugom) a da su im kosinusi jednaki.

$x=y=z$

$\cos x-\cos 3x=0 \Rightarrow \cos x-\cos 3x=(-2)\sin(-x)\sin(2x)=2\sin(x)\sin(2x)=0$

$$\left.\begin{array}{l} \cos x=\cos\frac{x-3x+x+3x}{2}=\cos\left(\frac{x-3x}{2}+\frac{x+3x}{2}\right)=\cos(-x)\cos(2x)-\sin(-x)\sin(2x) \\ \cos 3x=\cos\frac{x+3x-x+3x}{2}=\cos\left(\frac{x+3x}{2}-\frac{x-3x}{2}\right)=\cos(2x)\cos(-x)+\sin(2x)\sin(-x) \end{array}\right\}-$$

$\sin 2x=0$ ili $\sin x=0 \Rightarrow 2x=0$ ili $2x=\pi$ ili $x=0$ ili $\cancel{x=\pi}$

$0\le x\le\pi$ $\qquad x=0$ ili $x=\frac{\pi}{2}$ ili $x=\pi$

Stacionarne tačke su $M_1(0,0,0)$, $M_2(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ i $M_3(\pi,\pi,\pi)$.

$u''_{xx}=-\sin x+\sin(x+y+z)$ $\quad u''_{yx}=\sin(x+y+z)$ $\quad u''_{zx}=\sin(x+y+z)$

$u''_{xy}=\sin(x+y+z)$ $\quad u''_{yy}=-\sin y+\sin(x+y+z)$ $\quad u''_{zy}=\sin(x+y+z)$

$u''_{xz}=\sin(x+y+z)$ $\quad u''_{yz}=\sin(x+y+z)$ $\quad u''_{zz}=-\sin z+\sin(x+y+z)$

za tačku $M_1(0,0,0)$

$$T=\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$$ ispitujemo f-ju u okolini tačke M_1

$\Delta u=u(0+\varepsilon,0+\omega,0+\delta)-u(0,0,0)=\sin\varepsilon+\sin\omega+\sin\delta-\sin(\underbrace{\varepsilon+\omega}_{\alpha}+\underbrace{\delta}_{\beta})$

$=\sin\varepsilon+\sin\omega+\sin\delta-(\sin(\varepsilon+\omega)\cos\delta+\sin\delta\cos(\varepsilon+\omega))=$

$=\underline{\sin\varepsilon}+\underline{\sin\omega}+\sin\delta-\underline{\underline{\cos\delta}}(\sin\varepsilon\cos\omega-\underline{\underline{\sin\omega\cos\varepsilon}})$

$-\sin\delta(\cos\varepsilon\cos\omega-\sin\varepsilon\sin\omega)=\sin\varepsilon(1-\cos\delta\cos\omega)+\sin\omega(1-\cos\delta\cos\varepsilon)$

$+\sin\delta(1-\cos\varepsilon\cos\omega)+\sin\varepsilon\sin\omega\sin\delta\ge0 \quad \forall\varepsilon\ \forall\omega\ \forall\delta$

Ove iste vrijednosti dobijemo i za tačku $M_3(\pi,\pi,\pi)$

$\Rightarrow$ f-ja u tački M_1 i M_3 ima minimum, $\min=0$

za tačku $M_2(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$

$$T=\begin{bmatrix} -2&-1&-1\\-1&-2&-1\\-1&-1&-2 \end{bmatrix}$$

$a_{11}<0$, $\begin{vmatrix} -2&-1\\-1&-2 \end{vmatrix}=4-1=3>0$, $\det T=\begin{vmatrix} -2&-1&-1\\-1&-2&-1\\-1&-1&-2 \end{vmatrix}\begin{array}{l} I_v-III_v\cdot 2 \\ II_v-III_v \end{array}\begin{vmatrix} 0&1&3\\0&-1&1\\-1&-1&-2 \end{vmatrix}$

$\det T=-4 \Rightarrow$ U tački f-ja ima maksimum, $u_{max}=4$

#) Naći ekstreme f-je $u = x^2+y^2+z^2+2x+4y-6z$.

Rj. $u'_x = 2x+2$ $\quad$ $u'_y = 2y+4$ $\quad$ $u'_z = 2z-6$

$$\begin{aligned} 2x+2&=0\\ 2y+4&=0\\ 2z-6&=0 \end{aligned}$$
$x=-1,\ y=-2,\ z=3$

Tačka $M(-1,-2,3)$ je stacionarna tačka.

$u''_{xx}=2$ $\quad$ $u''_{yx}=0$ $\quad$ $u''_{zx}=0$

$u''_{xy}=0$ $\quad$ $u''_{yy}=2$ $\quad$ $u''_{zy}=0$

$u''_{xz}=0$ $\quad$ $u''_{yz}=0$ $\quad$ $u''_{zz}=2$

$$T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$a_{11}>0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 2 > 0$

i $\det T = 8 > 0 \Rightarrow$ u tački M f-ja ima minimum

$z_{min} = 1+4+9-2-8-18 = -14$

Uslovni ekstremi f-ja dviju promjenjivih

Ako trebamo naći ekstrem f-je $z=f(x,y)$ tako da x i y zadovoljavaju neki uslov $g(x,y)=0$ tada tražimo ekstrem Lagranžove f-je $F(x,y,\lambda) = f(x,y) + \lambda g(x,y)$.

$$\frac{\partial F}{\partial x}=0 \qquad \frac{\partial F}{\partial y}=0 \qquad \frac{\partial F}{\partial z}=0$$
SISTEM

rješavanjem sistema dobićemo neke stacionarne tačke i dalji proces se nastavlja kao kod traženja ekstrema f-je dvije promjenjive

II način: neka je $M(p_1,p_2)$ neka stacionarna tačka

$$d^2F(p_1,p_2) = F''_{xx}(p_1,p_2)\,dx^2 + 2F''_{xy}(p_1,p_2)\,dx\,dy + F''_{yy}(p_1,p_2)\,dy^2$$

$d^2F(p_1,p_2) > 0 \Rightarrow z_{min}(p_1,p_2)$

$d^2F(p_1,p_2) < 0 \Rightarrow z_{max}(p_1,p_2)$

Ako se desi slučaj da imamo više uslova, onda uvodimo više parametara $(\lambda, \mu, \dots)$.

1) Naći ekstreme f-je $z = 6-4x-3y$ uz uslov $x^2+y^2=1$.

Rj. $F(x,y) = 6-4x-3y+\lambda(x^2+y^2-1)$

$$\frac{\partial F}{\partial x} = -4+2\lambda x \qquad \frac{\partial F}{\partial y} = -3+2\lambda y \qquad \frac{\partial F}{\partial \lambda} = x^2+y^2-1$$

$$\begin{aligned} 2\lambda x - 4 &= 0\\ 2\lambda y - 3 &= 0\\ x^2+y^2-1&=0 \end{aligned} \qquad \begin{aligned} 2\lambda x &= 4\\ 2\lambda y &= 3\\ x^2+y^2&=1 \end{aligned}$$

$x = \frac{2}{\lambda}$, $y = \frac{3}{2\lambda}$, $x^2+y^2=1$

$\frac{4}{\lambda^2} + \frac{9}{4\lambda^2} = 1$

$\frac{25}{4\lambda^2} = 1$

$4\lambda^2 = 25$

$\lambda_{1,2} = \pm\frac{5}{2}$

$\lambda_1 = -\frac{5}{2} \Rightarrow x_1 = -\frac{4}{5}$; $y_1 = \frac{3}{2\cdot(-\frac{5}{2})} = -\frac{3}{5}$

$\lambda_2 = \frac{5}{2} \Rightarrow x_2 = \frac{2}{\frac{5}{2}} = \frac{4}{5}$; $y_2 = \frac{3}{2\cdot\frac{5}{2}} = \frac{3}{5}$

Stacionarne tačke su $M(-\frac{4}{5}, -\frac{3}{5})$ za $\lambda = -\frac{5}{2}$ i $N(\frac{4}{5}, \frac{3}{5})$ za $\lambda = \frac{5}{2}$.

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda \qquad \text{za } M\left(-\frac{4}{5}, -\frac{3}{5}\right),\ \lambda = -\frac{5}{2}$$

$A=-5,\ B=0,\ C=-5,\quad D=AC-B^2=25>0$

$$\frac{\partial^2 F}{\partial x \partial y} = 0 \qquad \text{f-ja ima ekstrem, } A<0 \text{ f-ja ima maksimum}$$

$$\frac{\partial^2 F}{\partial y^2} = 2\lambda \qquad z_{max}\left(-\frac{4}{5}, -\frac{3}{5}\right) = 6-4\left(-\frac{4}{5}\right)-3\left(-\frac{3}{5}\right) = \frac{30+16+9}{5} = \frac{55}{5} = 11$$

za $N\left(\frac{4}{5}, \frac{3}{5}\right)$, $\lambda = \frac{5}{2}$, $A=5,\ B=0,\ C=5,\ D=AC-B^2=25>0$

f-ja ima ekstrem u tački N, $A>0$ f-ja ima minimum

$$z_{min}\left(\frac{4}{5}, \frac{3}{5}\right) = 6-4\cdot\frac{4}{5}-3\cdot\frac{3}{5} = \frac{30-16-9}{5} = \frac{5}{5} = 1$$

2. Naći uslovne ekstreme f-je $z=y+2x+3$ uz uslov $x^2-6x+y+5=0$.

Rj. $F(x,y) = 2x+y+3+\lambda(x^2-6x+y+5)$

$$\frac{\partial F}{\partial x} = 2+2\lambda x-6\lambda \qquad \frac{\partial F}{\partial y} = 1+\lambda \qquad \frac{\partial F}{\partial \lambda} = x^2-6x+y+5$$

$2\lambda x-6\lambda+2=0 \quad |:2$

$\lambda+1=0$

$x^2-6x+y+5=0$

$\lambda x = 3\lambda-1$

$\lambda=-1$

$x^2-6x+y+5=0$

$-x=-3-1$

$x=4$

$x^2-6x+y+5=0$

$16-24+y+5=0$

$y=3$

Tačka $M(4,3)$ je stacionarna tačka, za $\lambda=-1$

$$\frac{\partial^2 F}{\partial x^2} = 2\lambda \qquad \frac{\partial^2 F}{\partial x \partial y} = 0 \qquad \frac{\partial^2 F}{\partial y^2} = 0$$

$M(4,3)$, $\lambda=-1$

$A=-2,\ B=0,\ C=0 \Rightarrow D=AC-B^2=0$

$$d^2F = \frac{\partial^2 F}{\partial x^2}dx^2 + \frac{\partial^2 F}{\partial x \partial y}\cancel{dxdy} + \frac{\partial^2 F}{\partial y^2}dy^2$$

$$d^2F = 2\lambda\, dx^2 \Rightarrow d^2F = -2\,dx^2 < 0$$

U tački $M(4,3)$ f-ja ima maksimum, $z_{max}(4,3) = 3+8+3 = 14$

3. Odrediti ekstreme f-je $z=x^2+y^2$ uz uslov $\frac{x}{2}+\frac{y}{3}=1$.

Rj. $z_{min}\left(\frac{18}{13}, \frac{12}{13}\right) = \frac{36}{13}$, $\lambda = -\frac{72}{13}$

4. Naći uslovne ekstreme f-je $z=\ln(x+y)$, ako je $x^2+2y^2=4$.

Rj. $z_{max}\left(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) = \ln\left(3\sqrt{\frac{2}{3}}\right)$, $\lambda = -\frac{1}{8}$.

#. Naći uslovne ekstreme f-je $z=2x^4+8y^4+24$ ako je $8x+4y=1$.

Rj. $F(x,y,\lambda) = 2x^4+8y^4+24+\lambda(8x+4y-1)$

$$\frac{\partial F}{\partial x} = 8x^3+8\lambda \qquad \frac{\partial F}{\partial y} = 32y^3+4\lambda \qquad \frac{\partial F}{\partial \lambda} = 8x+4y-1$$

$8x^3+8\lambda=0 \quad |:8$

$32y^3+4\lambda=0 \quad |:4$

$x^3+\lambda=0$

$-\ 8y^3+\lambda=0$

$x^3-8y^3=0$

$x^3=8y^3$

$x=2y$

$8x+4y-1=0$

$8\cdot 2y+4y-1=0$

$20y=1$

$y=\frac{1}{20}$

$x=2\cdot\frac{1}{20}=\frac{1}{10}$

$M_1\left(\frac{1}{10}, \frac{1}{20}\right)$ je stacionarna tačka

$$\frac{\partial^2 F}{\partial x^2} = 24x^2 \qquad \frac{\partial^2 F}{\partial x \partial y} = 0 \qquad \frac{\partial^2 F}{\partial y^2} = 96y^2$$

$D=AC-B^2$

$M_1\left(\frac{1}{10}, \frac{1}{20}\right)$

$A = 24\cdot\frac{1}{100} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$

$B=0$

$C = \cancel{96}\cdot\frac{1}{\cancel{20}\cdot\cancel{20}} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$

$D = \left(\frac{6}{25}\right)^2 > 0$ f-ja ima ekstrem

$A>0$ f-ja ima minimum

$$z_{min}\left(\frac{1}{10}, \frac{1}{20}\right) = 2\cdot\frac{1}{10^4}+8\cdot\frac{1}{20^4}+24 = \frac{2}{10\,000}+\frac{\cancel{8}}{\cancel{20}\cdot\cancel{20}\cdot\cancel{20}\cdot 20}+24$$

$$= \frac{2}{10\,000}+\frac{1}{20\,000}+\frac{24\,000}{1\,000} = \frac{4+1+480\,000}{20\,000} =$$

$$= \frac{480\,005}{20\,000}\ \frac{:5}{:5} = \frac{96\,001}{4\,000}$$

$z_{min} = \frac{96\,001}{4\,000}$ je minimum f-je u tački $M\left(\frac{1}{10}, \frac{1}{20}\right)$

Naći uslovne ekstreme f-je $z=(x-y)^4+1$ ako je $x^2+y^2=18$.

Rj. $F(x,y,\lambda)=(x-y)^4+1+\lambda(x^2+y^2-18)$

$$\frac{\partial F}{\partial x}=4(x-y)^3+2\lambda x$$
$$\frac{\partial F}{\partial y}=4(x-y)^3\cdot(-1)+2\lambda y$$
$$\frac{\partial F}{\partial \lambda}=x^2+y^2-18$$

$$4(x-y)^3+2\lambda x=0 \quad \ldots(1)$$
$$-4(x-y)^3+2\lambda y=0 \quad \ldots(2)$$
$$x^2+y^2-18=0 \quad \ldots(3)$$

(1) (2): $2\lambda x+2\lambda y=0 \quad /:2$

$\lambda(x+y)=0$

$\lambda=0$ ili $x+y=0$

a) $x+y=0$

$x=-y$

$(-y)^2+y^2=18$

$2y^2=18$

$y_{1,2}=\pm 3$

$y_1=-3 \Rightarrow x_1=3$

$y_2=3 \Rightarrow x_2=-3$

$M_1(3,-3)$, $M_2(-3,3)$

za M_1 (1) $\Rightarrow 6\lambda=-4\cdot 6^3$

za M_2 (1) $\Rightarrow -6\lambda=-4\cdot(-6)^3$

$\Rightarrow \lambda=-144$

b) $\lambda=0$

(1) $\Rightarrow 4(x-y)^3=0$

$x=y$

$2y^2=18$

$y_{3,4}=\pm 3$

$M_3(-3,-3)$

$M_4(3,3)$

$\lambda=0$

Stacionarne tačke su $M_1(3,-3)$ $M_2(-3,3)$ za $\lambda=-144$ i $M_3(-3,-3)$ i $M_4(3,3)$ za $\lambda=0$.

$$\frac{\partial^2 F}{\partial x^2}=12(x-y)^2+2\lambda$$
$$\frac{\partial F}{\partial x\partial y}=-12(x-y)^2$$
$$\frac{\partial F}{\partial y^2}=12(x-y)^2+2\lambda$$

$D=AC-B^2$

$M_1(3,-3)$, $\lambda=-144$

$A=12\cdot 36-2\cdot 144=144$

$B=-12\cdot 36=-432$

$C=144$

$D=20\,736-186\,624$

f-ja u tački M_1 nema ekstrema

$M_2(-3,3)$, $\lambda=-144$

$\left.\begin{array}{l} A=144 \\ B=432 \\ C=144 \end{array}\right\} D<0$

f-ja u tački M_2 nema ekstrem

$M_3(-3,-3)$, $\lambda=0$

$A=0$, $B=0$, $C=0$ $\Rightarrow$ $D=0$ potrebno je ispitati f-ju u okolini tačke $M_3(-3,-3)$

$\Delta z(M_3)=z(-3+\varepsilon,-3+\omega)-z(-3,-3)=(-3+\varepsilon+3-\omega)^4+1-1=(\varepsilon-\omega)^4>0 \quad \forall\varepsilon$ i $\forall\omega$

Priraštaj f-je u okolini tačke M_3 je pozitivan pa f-ja u M_3 ima minimum, $z_{min}(-3,-3)=1$

$M_4(3,3)$, $\lambda=0$

$A=0$, $B=0$, $C=0$ $\Rightarrow$ $D=0$ potrebno je ispitati f-ju u okolini tačke $M_4(3,3)$

$\Delta z(M_4)=z(3+\varepsilon,3+\omega)-z(3,3)=(3+\varepsilon-3-\omega)^4+1-1=(\varepsilon-\omega)^4>0 \quad \forall\varepsilon$ i $\forall\omega$

Priraštaj f-je u okolini tačke M_4 je pozitivan pa f-ja u M_4 ima minimum, $z_{min}(3,3)=1$.

Naći uslovne ekstreme f-je $z=2x+4y$ ako je $\frac{2}{x}+\frac{4}{y}=3$.

Rj. Formirajmo Lagranžovu f-ju $F(x,y,\lambda)=2x+4y+\lambda\left(\frac{2}{x}+\frac{4}{y}-3\right)$.

$$\frac{\partial F}{\partial x}=2+2\lambda\cdot\frac{(-1)}{x^2} \qquad \left(\frac{1}{x}\right)'=(x^{-1})'=(-1)(x^{-2})$$
$$\frac{\partial F}{\partial y}=4+4\lambda\cdot\frac{(-1)}{y^2} \qquad (x^{-2})'=(-2)x^{-3}=\frac{-2}{x^3}$$
$$\frac{\partial F}{\partial \lambda}=\frac{2}{x}+\frac{4}{y}-3$$

Formirajmo sistem

$$4-\frac{4\lambda}{y^2}=0 \quad /:4$$
$$2-\frac{2\lambda}{x^2}=0 \quad /:2$$
$$\frac{2}{x}+\frac{4}{y}=3$$

$1-\frac{\lambda}{x^2}=0 \qquad 1=\frac{\lambda}{x^2} \quad (1)$

$1-\frac{\lambda}{y^2}=0 \qquad 1=\frac{\lambda}{y^2} \quad (2)$

$\frac{2}{x}+\frac{4}{y}=3 \qquad \frac{2}{x}+\frac{4}{y}=3 \quad (3)$

(1) i (2) $\Rightarrow \frac{\lambda}{x^2}=\frac{\lambda}{y^2} \Rightarrow x^2=y^2$

tj. $x=\pm y$

za $x=y$ iz (3) $\frac{2}{x}+\frac{4}{x}=3$

$\frac{6}{x}=3 \Rightarrow x=2 \Rightarrow y=2$

za $x=-y$ iz (3)

$\frac{2}{x}-\frac{4}{x}=3 \Rightarrow -\frac{2}{x}=3$

$3x=-2 \Rightarrow x=-\frac{2}{3}$

$\Rightarrow y=\frac{2}{3}$

za $M_1(2,2) \Rightarrow 2-2\lambda\cdot\frac{1}{4}=0$

$\lambda=4$

za $M_2\left(-\frac{2}{3},\frac{2}{3}\right) \Rightarrow 2-2\lambda\cdot\frac{9}{4}=0 \Rightarrow \lambda=\frac{4}{9}$

Stacionarne tačke su $M_1(2,2)$ za $\lambda=4$ i $M_2\left(-\frac{2}{3},\frac{2}{3}\right)$ za $\lambda=\frac{4}{9}$.

$$\frac{\partial^2 F}{\partial x^2}=\frac{4\lambda}{x^3}$$
$$\frac{\partial^2 F}{\partial x\partial y}=0$$
$$\frac{\partial^2 F}{\partial y^2}=\frac{8\lambda}{y^3}$$

Za $M_1(2,2)$, $\lambda=4$

$A=\frac{16}{8}=2$, $B=0$, $C=\frac{32}{8}=4$, $D=AC-B^2=8>0$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2)=4+8=12$

Za $M_2\left(-\frac{2}{3},\frac{2}{3}\right)$, $\lambda=\frac{4}{9}$, $A=\frac{\frac{16}{9}}{-\frac{8}{27}}=-\frac{16\cdot 27}{8\cdot 9}=-2\cdot 3=-6$

$B=0$, $C=\frac{\frac{32}{9}}{\frac{8}{27}}=\frac{32\cdot 27}{8\cdot 9}=4\cdot 3=12$, $D=AC-B^2=-72<0 \Rightarrow$

$\Rightarrow$ f-ja u tački M_2 nema ekstremnu vrijednost

Nađi uslovne ekstreme f-je $z=xy$ ako je $x^2+y^2=2ax$, $a>0$.

Rj: Posmatramo f-ju $F(x,y,\lambda)=xy+\lambda(x^2+y^2-2ax)$

$$\frac{\partial F}{\partial x}=y+2\lambda x-2a\lambda$$

$$\frac{\partial F}{\partial y}=x+2\lambda y$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-2ax$$

$$\begin{aligned}
&y+2\lambda x-2a\lambda=0\\
&x+2\lambda y=0\\
&\underline{x^2+y^2-2a\cdot x=0}\\
(1)\quad &y+2\lambda(x-a)=0 \Rightarrow x-a=\frac{-y}{2\lambda} \quad \dots(*)\\
(2)\quad &x=-2\lambda y\\
(3)\quad &\underline{x^2-2\cdot x\cdot a+a^2-a^2+y^2=0}
\end{aligned}$$

$$\begin{aligned}
(2)\text{ u }(1):\quad &y+2\lambda(-2\lambda y-a)=0\\
(3):\quad &\underline{(x-a)^2+y^2=a^2}\\
&y-4\lambda^2y-2a\lambda=0\\
&y(1-4\lambda^2)=2a\lambda\\
&y=\frac{2a\lambda}{1-4\lambda^2}
\end{aligned}$$

(*) u (3):

$$\frac{y^2}{4\lambda^2}+y^2=a^2$$

$$y^2\left(\frac{1}{4\lambda^2}+1\right)=a^2$$

$$y^2\left(\frac{1+4\lambda^2}{4\lambda^2}\right)=a^2$$

$$y^2=\frac{4a^2\lambda^2}{1+4\lambda^2}$$

$$y=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}}$$

$$y=\frac{2a\lambda}{1-4\lambda^2}=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}} \Rightarrow 1-4\lambda^2=\pm\sqrt{1+4\lambda^2}$$

$$(1-4\lambda^2)^2=1+4\lambda^2$$

$$16\lambda^4-8\lambda^2+1=1+4\lambda^2$$

$$16\lambda^4-12\lambda^2=0$$

$$\lambda^2(16\lambda^2-12)=0 \Rightarrow \lambda_1=0,\quad \lambda_{2,3}=\pm\sqrt{\frac{12}{16}}=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt3}{2}$$

$\lambda_1=0$: $y=0$, $x=0$

$\lambda_2=\frac{\sqrt3}{2}$:

$$\begin{aligned}
&y+\sqrt3x-a\sqrt3=0\\
&x+y\sqrt3=0 \quad /\cdot\sqrt3\\
&\underline{x^2+y^2-2ax=0}
\end{aligned}
\qquad
\begin{aligned}
&\sqrt3x+y=a\sqrt3\\
-\ &\underline{\sqrt3x+3y=0}\\
&-2y=a\sqrt3\\
&y=\frac{-a}{2}\sqrt3 \qquad x=\frac{3}{2}a
\end{aligned}$$

$\lambda_3=-\frac{\sqrt3}{2}$:

$$\begin{aligned}
&y-x\sqrt3+a\sqrt3=0\\
&x-y\sqrt3=0 \quad /\cdot\sqrt3\\
&\underline{x^2+y^2-2ax=0}
\end{aligned}
\qquad
\begin{aligned}
&-x\sqrt3+y=-a\sqrt3\\
+\ &\underline{x\sqrt3-3y=0}\\
&-2y=-a\sqrt3\\
&y=\frac{\sqrt3}{2}a \Rightarrow x=\frac{3}{2}a
\end{aligned}$$

Stacionarne tačke su $M_1(0,0)$ za $\lambda=0$, $M_2(\frac{3}{2}a, -\frac{\sqrt3}{2}a)$ za $\lambda=\frac{\sqrt3}{2}$; $M_3(\frac{3}{2}a, \frac{\sqrt3}{2}a)$ za $\lambda=-\frac{\sqrt3}{2}$.

$$\frac{\partial^2F}{\partial x^2}=2\lambda \qquad \frac{\partial^2F}{\partial x\partial y}=1 \qquad \frac{\partial^2F}{\partial y^2}=2\lambda$$

$M_1(0,0)$, $\lambda=0$

$D=AC-B^2=-1<0 \Rightarrow$ f-ja u tački $M_1(0,0)$ nema ekstrem

$M_2(\frac{3}{2}a, -\frac{\sqrt3}{2}a)$, $\lambda=\frac{\sqrt3}{2}$

$D=AC-B^2=3-1=2>0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A=\sqrt3>0 \Rightarrow$ f-ja ima minimum

$$z_{min}\left(\frac{3}{2}a, -\frac{\sqrt3}{2}a\right)=-\frac{3\sqrt3}{4}a^2$$

$M_3(\frac{3}{2}a, \frac{\sqrt3}{2}a)$ za $\lambda=-\frac{\sqrt3}{2}$

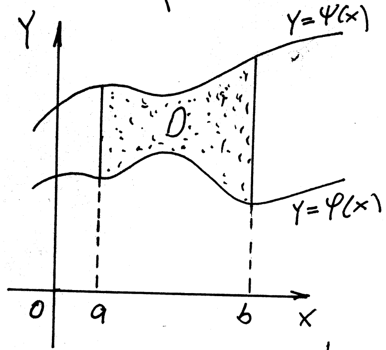
$D=AC-B^2=3-1>0 \Rightarrow$ f-ja ima ekstrem

$A=-\sqrt3<0 \Rightarrow$ f-ja u tački M_3 ima maksimum

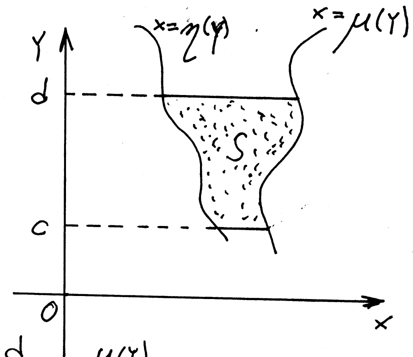
$$z_{max}\left(\frac{3}{2}a, \frac{\sqrt3}{2}a\right)=\frac{3\sqrt3}{4}a^2$$

Dvostruki integral

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right] dx$$



D – oblast integracije



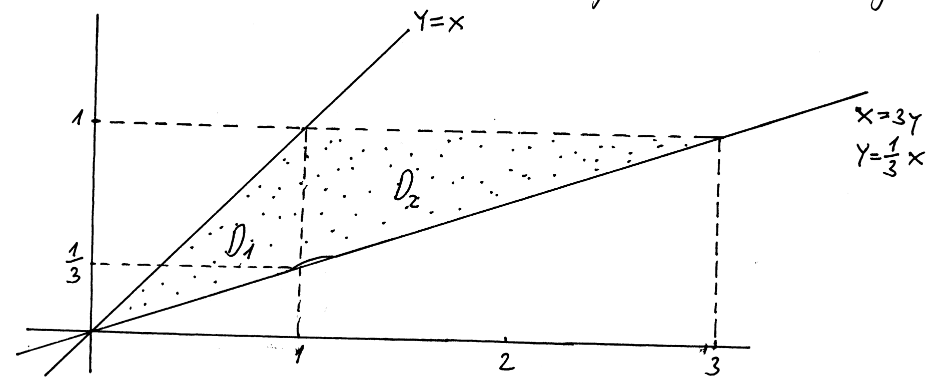
$$\iint_S f(x,y)\,dx\,dy = \int_c^d dy \int_{\eta(y)}^{\mu(y)} f(x,y)\,dx = \int_c^d \left[\int_{\eta(y)}^{\mu(y)} f(x,y)\,dx \right] dy$$

(#) Izmeniti poredak integracije u integralu

$$I = \int_0^1 dy \int_y^{3y} f(x,y)\,dx .$$

Rj.

$x=3y$; $x=y$ su prave. Skicirajmo oblast integracije



$$D: \begin{cases} 0 \le y \le 1 \\ y \le x \le 3y \end{cases} = D_1 \cup D_2$$

$$D_1 : \begin{cases} 0 \le x \le 1 \\ \frac{1}{3}x \le y \le x \end{cases}$$

$$D_2 : \begin{cases} 1 \le x \le 3 \\ \frac{1}{3}x \le y \le 1 \end{cases}$$

$$\int_0^1 dy \int_y^{3y} f(x,y)\,dx = \int_0^1 dx \int_{\frac{1}{3}x}^{x} f(x,y)\,dy + \int_1^3 dx \int_{\frac{1}{3}x}^{1} f(x,y)\,dy$$

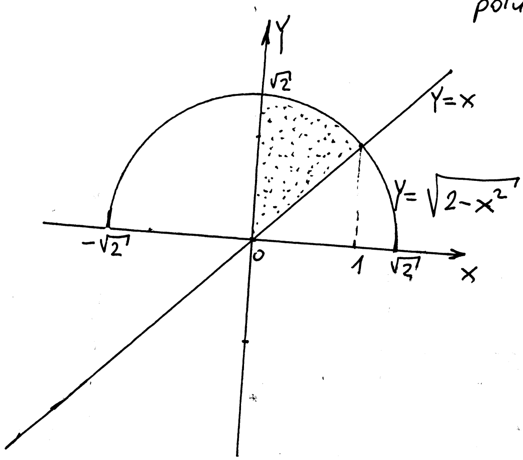
Izmeniti poredak integracije u integralu

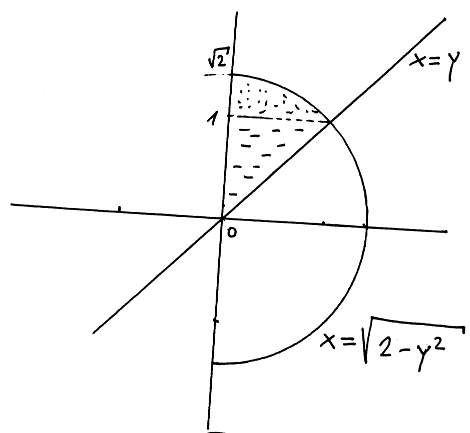
$$I=\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x,y)\,dy .$$

Rj. $y=x$ prava $\qquad y^2=2-x^2$

$y=\sqrt{2-x^2}$ parabola $\qquad x^2+y^2=2$

krug sa centrom u tački $(0,0)$ poluprečnika $r=\sqrt{2}\approx 1{,}41$



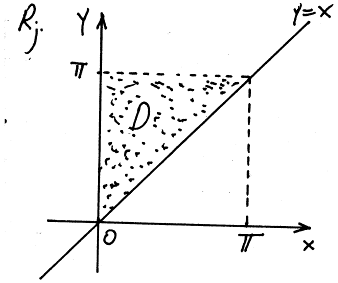


$$I=\int_0^1 dx\int_x^{\sqrt{2-x^2}} f(x,y)\,dy=\int_0^1 dy\int_0^{y} f(x,y)\,dx+\int_1^{\sqrt{2}} dy\int_0^{\sqrt{2-y^2}} f(x,y)\,dx$$

Izračunati dvostruki integral $\iint_D \cos(x+y)\,dx\,dy$

ako je $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le \pi \ ; \ x\le y\le \pi\}$.

Rj.



$$\iint_D \cos(x+y)\,dx\,dy=\int_0^\pi dx\int_x^\pi \cos(x+y)\,dy=$$

$$=\int_0^\pi dx\,\sin(x+y)\Big|_x^\pi=\int_0^\pi[\sin(x+\pi)-\sin 2x]\,dx \overset{(*)}{=}$$

$\sin(x+\pi)=\sin x\cos\pi+\sin\pi\cos x=-\sin x$

$$\overset{(*)}{=}\int_0^\pi(-\sin x-\sin 2x)\,dx=-\int_0^\pi \sin x\,dx-\int_0^\pi \sin 2x\,dx=\cos x\Big|_0^\pi+\frac{1}{2}\cos 2x\Big|_0^\pi=$$

$$=(-1-1)+\frac{1}{2}(1-1)=-2$$

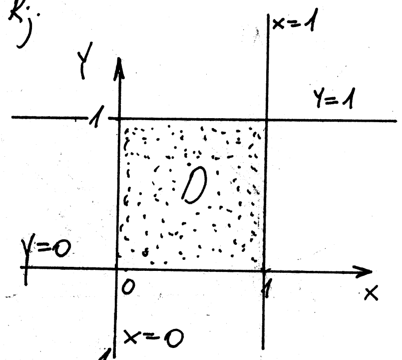
II način

$$\iint_D \cos(x+y)\,dx\,dy=\int_0^\pi dy\int_0^y \cos(x+y)\,dx=\int_0^\pi dy\,\sin(x+y)\Big|_0^y=$$

$$\int_0^\pi(\sin 2y-\sin y)\,dy=-\frac{1}{2}\cos 2y\Big|_0^\pi+\cos y\Big|_0^\pi=-\frac{1}{2}(1-1)+(-1-1)=-2$$

Izračunati vrijednost integrala $I=\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy$ gdje je $D=\{(x,y)\in R^2 \mid 0\le x\le 1 \text{ i } 0\le y\le 1\}$.

Rj.



I način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy=\int_0^1 dx\int_0^1 \frac{x^2}{1+y^2}\,dy = \int_0^1 x^2\,dx\int_0^1\frac{dy}{1+y^2}=\int_0^1 x^2 \operatorname{arctg} y\Big|_0^1 dx=$$

$$= \frac{\pi}{4}\int_0^1 x^2\,dx=\frac{\pi}{4}\cdot\frac{x^3}{3}\Big|_0^1=\frac{\pi}{12}$$

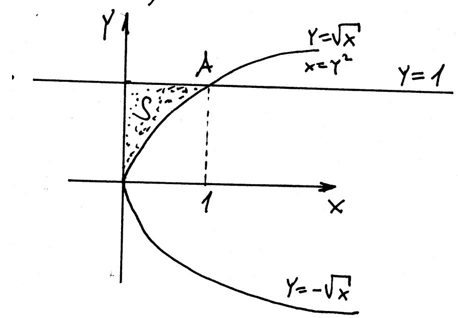
II način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy = \int_0^1 dy\int_0^1 \frac{x^2}{1+y^2}\,dx = \int_0^1\frac{dy}{1+y^2}\int_0^1 x^2\,dx$$

$$= \int_0^1 \frac{1}{1+y^2}\cdot\frac{x^3}{3}\Big|_0^1 dy = \frac13\int_0^1\frac{dy}{1+y^2}=\frac13 \operatorname{arctg} x\Big|_0^1=\frac13\cdot\frac{\pi}{4}=\frac{\pi}{12}$$

Izračunati integral $\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy$, gdje je S oblast omeđena parabolom $y^2=x$, te pravama $x=0$, $y=1$.

Rj. Nacrtajmo sliku



Tačka $A(1,1)$ je presjek parabole $y^2=x$ i prave $y=1$.

Moguća su dva načina:

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy = \int_0^1\left[\int_0^{y^2} e^{-\frac{x}{y}}\,dx\right]dy$$

ili

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy = \int_0^1\left[\int_{\sqrt{x}}^1 e^{-\frac{x}{y}}\,dy\right]dx$$

Kako $\int e^{-\frac{1}{t}}\,dt=?$ imamo:

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy = \int_0^1\left[\int_0^{y^2} e^{-\frac{x}{y}}\,dx\right]dy = \int_0^1 \left(-y\, e^{-\frac{x}{y}}\Big|_0^{y^2}\right)dy = -\int_0^1 y\left(e^{-y}-1\right)dy \overset{(*)}{=}$$

$$\int e^{-\frac{x}{y}}\,dx = \left|\begin{matrix}-\frac{x}{y}=t\\ -\frac{1}{y}dx=dt\end{matrix}\right| = \int e^t(-y)\,dt=-y\,e^t+c=-y\,e^{-\frac{x}{y}}+c$$

$$\overset{(*)}{=} \int_0^1\left(y-y\,e^{-y}\right)dy = \int_0^1 y\,dy+\int_0^1(-y)\,e^{-y}\,dy = \frac12 y^2\Big|_0^1 + (y+1)\,e^{-y}\Big|_0^1 \overset{(\Delta)}{=}$$

$$\int t\,e^t\,dt = \left|\begin{matrix}u=t & dv=e^t dt\\ du=dt & v=e^t\end{matrix}\right| = t\,e^t-\int e^t\,dt=(t-1)\,e^t+c$$

$$\int(-t)\,e^{-t}\,dt = \left|\begin{matrix}u=-t & dv=e^{-t} dt\\ du=-dt & v=-e^{-t}\end{matrix}\right| = t\,e^{-t}-\int e^{-t}\,dt=(t+1)\,e^{-t}+c$$

$$\overset{(\Delta)}{=} \frac12+\left(2e^{-1}-1\right)=\frac{2}{e}-\frac12$$

\# Izračunati dvostruki integral $\iint\limits_D (x^2+y)\,dx\,dy$

gdje je D površ ograničena linijama $y=x^2$ i $y^2=x$.

Rj. Nađimo presječnu tačku i nacrtajmo ove dvije krive

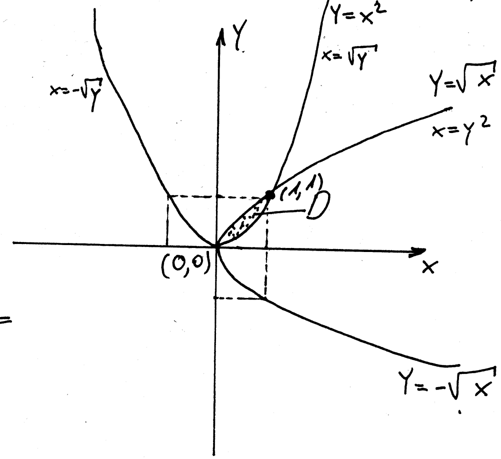
$$\begin{aligned} y&=x^2\\ y^2&=x\\ \hline x^4&=x \end{aligned}$$

$$x(x^3-1)=0$$

$$x(x-1)(x^2+x+1)=0$$

$$x=0 \text{ ili } x=1$$

Presječne tačke krivih su $(0,0)$ i $(1,1)$.



$$\iint\limits_D (x^2+y)\,dx\,dy = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2+y)\,dy =$$

$$= \int_0^1 dx \left(x^2 y\Big|_{x^2}^{\sqrt{x}} + \frac{1}{2}y^2\Big|_{x^2}^{\sqrt{x}} \right) = \int_0^1 \left[x^2(\sqrt{x}-x^2) + \frac{1}{2}(x-x^4)\right] dx$$

$$= \int_0^1 \left(x^2\sqrt{x} - x^4 + \frac{1}{2}x - \frac{1}{2}x^4 \right) dx = \int_0^1 \left(-\frac{3}{2}x^4 + x^{\frac{5}{2}} + \frac{1}{2}x \right) dx =$$

$$= -\frac{3}{2}\cdot\frac{1}{5}x^5\Big|_0^1 + \frac{2}{7}\cdot x^{\frac{7}{2}}\Big|_0^1 + \frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^1 = -\frac{3}{10} + \frac{2}{7} + \frac{1}{4} = \frac{-3\cdot 14+2\cdot 20+1\cdot 35}{140}$$

$$= \frac{-42+40+35}{140} = \frac{33}{140}$$

II način: $\iint\limits_D (x^2+y)\,dx\,dy = \int_0^1 \left(\int_{y^2}^{\sqrt{y}} (x^2+y)\,dx \right) dy = \ldots = \int_0^1 \left(\frac{4}{3}\sqrt{y^3} - y^3 - \frac{1}{3}y^6 \right) dy$

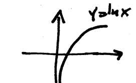
za vježbu

$$= \ldots = \frac{33}{140}$$

\# Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$,

gdje je D: $y=\ln x$, $x=2$, $x+y=1$.

Rj. Kriva $y=\ln x$ izgleda ovako



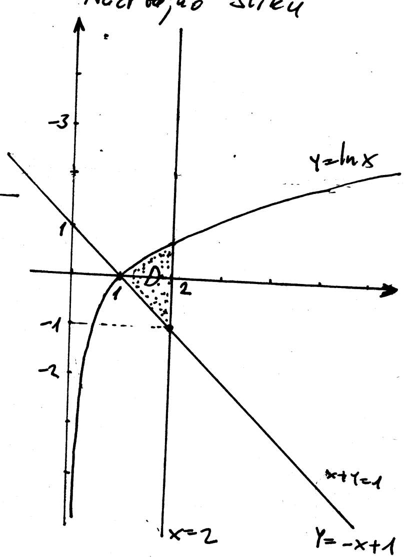
Pronađimo presječne tačke datih krivi.

$$\begin{aligned} y&=\ln x\\ x&=2\\ \hline y&=\ln 2 \approx 0{,}69 \end{aligned} \qquad (2, \ln 2)$$

$$\begin{aligned} y&=\ln x\\ x+y&=1\\ \hline y&=\ln x\\ y&=-x+1\\ \hline \ln x&=-x+1\\ x&=1 \end{aligned} \qquad (1,0)$$

$$\begin{aligned} x&=2\\ x+y&=1\\ \hline 2+y&=1\\ y&=-1 \end{aligned} \qquad (2,-1)$$

Nacrtajmo sliku



$$I=\iint\limits_D xy\,dx\,dy = \int_1^2 dx \int_{-x+1}^{\ln x} xy\,dy = \int_1^2 x\,dx \int_{-x+1}^{\ln x} y\,dy =$$

$$= \int_1^2 x\left(\frac{1}{2}y^2\Big|_{-x+1}^{\ln x}\right) dx = \frac{1}{2}\int_1^2 x\Big(\ln^2 x - \underbrace{(-x+1)^2}_{x^2-2x+1}\Big)\,dx =$$

$$= \frac{1}{2}\int_1^2 x\ln^2 x\,dx - \frac{1}{2}\int_1^2 (x^3-2x^2+x)\,dx$$

$$\int_1^2 x\ln^2 x\,dx = \left| \begin{matrix} u=\ln^2 x & dv=x\,dx \\ du=2\ln x\cdot\frac{1}{x}\,dx & v=\frac{1}{2}x^2 \end{matrix} \right| = \overbrace{\frac{1}{2}x^2\ln^2 x\Big|_1^2}^{\frac{1}{2}4(\ln 2)^2 \;=\; 2\cdot 0{,}48 \,\approx\, 0{,}96} - \int_1^2 x\ln x\,dx =$$

$$= \left| \begin{matrix} u=\ln x & dv=x\,dx \\ du=\frac{1}{x}\,dx & v=\frac{1}{2}x^2 \end{matrix} \right| = 2\ln^2 2 - \Big[\overbrace{\frac{1}{2}x^2\ln x\Big|_1^2}^{2\ln 2} - \frac{1}{2}\underbrace{\int_1^2 x\,dx}_{\frac{1}{2}x^2|_1^2 \;=\; \frac{1}{2}\cdot 3}\Big] = 2\ln^2 2 - 2\ln 2 + \frac{3}{4}$$

$$\int_1^2 (x^3-2x^2+x)\,dx = \frac{1}{4}x^4\Big|_1^2 - \frac{2}{3}x^3\Big|_1^2 + \frac{1}{2}x^2\Big|_1^2 =$$

$$= \frac{15}{4} - \frac{14}{3} + \frac{3}{2} = \frac{45-56+18}{12} = \frac{7}{12}$$

traženo rješenje

$$I = \frac{1}{2}\left(2\ln^2 2 - 2\ln 2 + \frac{3}{4}\right) - \frac{1}{2}\cdot\frac{7}{12} = \ln^2 2 - \ln 2 + \frac{3}{8} - \frac{7}{24} = \ln^2 2 - \ln 2 + \frac{1}{12}$$

Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$, gdje je D oblast ograničena linijama $xy=1$, $x+y=\frac{5}{2}$.

Rj. Skiciramo oblast D

$xy=1$

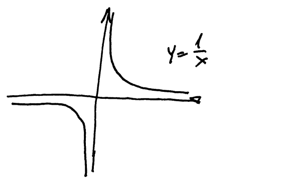
$y=\frac{1}{x}$ $\quad D: x\in\mathbb{R}\setminus\{0\}$

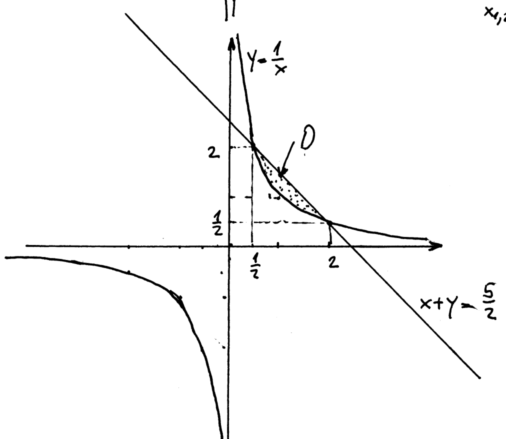
f-ja je neparna (simetrična u odnosu na 0)

ne siječe y-osu, ne siječe x-osu

$\lim\limits_{x\to 0+}\frac{1}{x}=+\infty$

$\lim\limits_{x\to\infty}\frac{1}{x}=0$





Nađimo presječne tačke krive $xy=1$ i prave $x+y=\frac{5}{2}$.

$$\begin{array}{l} xy=1\\ x+y=\frac{5}{2}\\ \hline y=\frac{1}{x}\\ x+y=\frac{5}{2}\\ \hline x+\frac{1}{x}=\frac{5}{2}\quad /\cdot x\end{array}$$

$x^2-\frac{5}{2}x+1=0 \quad /\cdot 2$

$2x^2-5x+2=0$

$D=25-16=9$

$x_{1,2}=\frac{5\pm 3}{4}$ $\quad x_1=\frac{2}{4}=\frac{1}{2}$

$x_2=2$

$x_1=\frac{1}{2}\Rightarrow y_1=2$

$x_2=2\Rightarrow y=\frac{1}{2}$

$$D:\begin{cases}\frac{1}{2}<x<2\\ \frac{1}{x}<y<\frac{5}{2}-x\end{cases}$$

$$\iint\limits_D xy\,dx\,dy=\int_{\frac{1}{2}}^{2}x\,dx\int_{\frac{1}{x}}^{\frac{5}{2}-x}y\,dy=$$

$$=\int_{\frac{1}{2}}^{2}x\,\frac{1}{2}y^2\Big|_{\frac{1}{x}}^{\frac{5}{2}-x}dx=\frac{1}{2}\int_{\frac{1}{2}}^{2}x\left(\left(\frac{5}{2}-x\right)^2-\frac{1}{x^2}\right)dx=\frac{1}{2}\int_{\frac{1}{2}}^{2}x\left(\frac{25}{4}-5x+x^2-\frac{1}{x^2}\right)dx$$

$$\frac{1}{2}\int_{\frac{1}{2}}^{2}\left(\frac{25}{4}x-5x^2+x^3-\frac{1}{x}\right)dx=\int_{\frac{1}{2}}^{2}\left(\frac{1}{2}x^3-\frac{5}{2}x^2+\frac{25}{8}x-\frac{1}{2}\cdot\frac{1}{x}\right)dx=\ldots=\frac{165}{128}-\ln 2$$

rješenje

Promjeniti poredak integracije u integralu

$$I=\int_{-7}^{1}dy\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}}f(x,y)\,dx$$

Rj.

Ako sa D označimo oblast integracije imamo

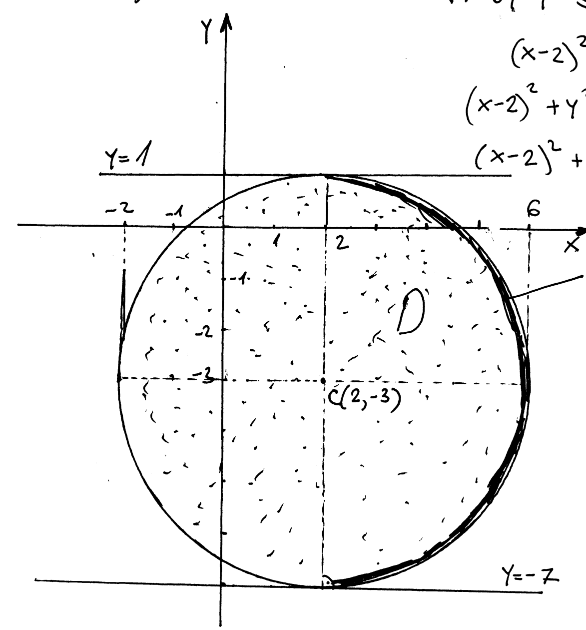
$$D:\begin{cases}2-\sqrt{7-6y-y^2}\le x\le 2+\sqrt{7-6y-y^2}\\ -7\le y\le 1.\end{cases}$$

Posmatrajmo $\overset{\text{prvo}}{}$ samo $\overset{\text{promjenjivu}}{}$ x: $\quad -\sqrt{7-6y-y^2}\le x-2\le\sqrt{7-6y-y^2}$

$(x-2)^2=7-6y-y^2$

$(x-2)^2+y^2+2\cdot 3\cdot y+9-9=7$

$(x-2)^2+(y+3)^2=16$ krug sa centrom u tački $C(2,-3)$ poluprečnika $r=4$



$(y+3)^2=16-\underbrace{(x-2)^2}_{x^2-4x+4}$

$(y+3)^2=12-x^2+4x$

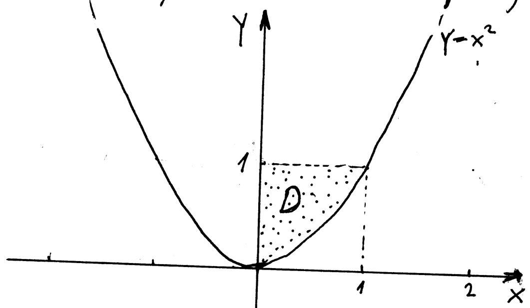
$y_{1,2}=-3\pm\sqrt{12-x^2+4x}$

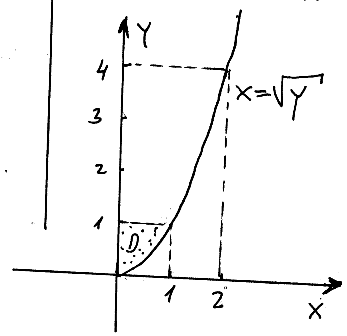
Prema tome

$$I=\int_{-7}^{1}dy\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}}f(x,y)\,dx=\int_{-2}^{6}dx\int_{-3-\sqrt{12-x^2+4x}}^{-3+\sqrt{12-x^2+4x}}f(x,y)\,dx\,.$$

Izračunati integral $I=\int_0^1 x^5 dx \int_{x^2}^1 e^{y^2} dy$.

Rj. Skicirajmo oblast integracije D





$$I=\int_0^1 x^5 dx\int_{x^2}^1 e^{y^2}dy = \iint_D x^5 e^{y^2}\, dx\, dy = \int_0^1 e^{y^2}dy\int_0^{\sqrt{y}} x^5 dx =$$

$$= \int_0^1 e^{y^2}\frac{1}{6} x^6\Big|_0^{\sqrt{y}} dy =$$

$$=\frac{1}{6}\int_0^1 e^{y^2}y^3\, dy = \left| \begin{matrix} u=y^2 & dv=e^{y^2} y\, dy=\frac{1}{2} e^{y^2}d(y^2) \\ du=2y\, dy & v=\frac{1}{2} e^{y^2} \end{matrix}\right| =$$

$$=\frac{1}{12}y^2e^{y^2}\Big|_0^1-\frac{1}{6}\int_0^1 e^{y^2}y\, dy = \frac{1}{12}(1\cdot e^1-0)-\frac{1}{12}\int_0^1 e^{y^2}d(y^2)=$$

$$=\frac{1}{12}e-\frac{1}{12}e^{y^2}\Big|_0^1=\frac{1}{12}e-\frac{1}{12}(e-1)=\frac{1}{12}e-\frac{1}{12}e+\frac{1}{12}=\frac{1}{12}$$

traženo rješenje

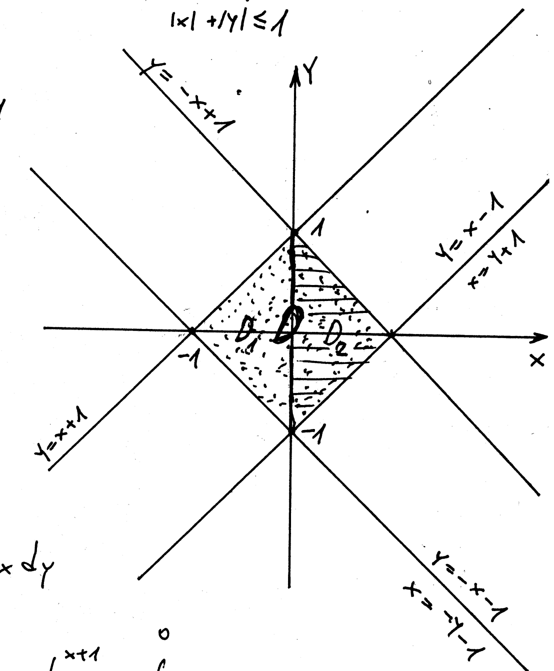
Izračunati vrijednost integrala $I=\iint_{|x|+|y|\le 1} x^2\, dx\, dy$

Rj.: $x<0,\ y<0 \Rightarrow -x-y\le 1$, $y\ge -x-1$

$x<0,\ y>0 \Rightarrow -x+y\le 1$, $y\le x+1$

$x\ge 0,\ y<0 \Rightarrow x-y\le 1$, $y\ge x-1$

$x\ge 0,\ y\ge 0 \Rightarrow x+y\le 1$, $y\le -x+1$



$$I=\iint_{|x|+|y|\le 1} x^2\, dx\, dy = \iint_{D_1} x^2\, dx\, dy + \iint_{D_2} x^2\, dx\, dy$$

$$\iint_{D_1} x^2\, dx\, dy = \int_{-1}^0 dx\int_{-x-1}^{x+1} x^2\, dy = \int_{-1}^0 x^2\, dx\ \underbrace{y\Big|_{-x-1}^{x+1}}_{x+1+x+1} = \int_{-1}^0 x^2(2x+2)\, dx =$$

$$=\int_{-1}^0 (2x^3+2x^2)\, dx = \frac{2}{4}x^4\Big|_{-1}^0 + \frac{2}{3}x^3\Big|_{-1}^0 = \frac{1}{2}\cdot(-1)+\frac{2}{3}\cdot 1 = \frac{2}{3}-\frac{1}{2}=\frac{1}{6}$$

$$\iint_{D_2} x^2\, dx\, dy = \int_0^1 dx\int_{x-1}^{-x+1} x^2\, dy = \int_0^1 x^2\, dx\ \underbrace{y\Big|_{x-1}^{-x+1}}_{-x+1-x+1} = \int_0^1 x^2(-2x+2)\, dx =$$

$$\int_0^1 (-2x^3+2x^2)\, dx = -\frac{2}{4}x^4\Big|_0^1 + \frac{2}{3}x^3\Big|_0^1 = -\frac{1}{2}+\frac{2}{3} = \frac{1}{6},$$

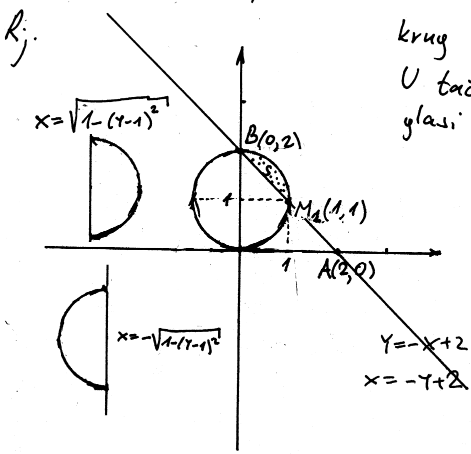
$$I=\iint_{|x|+|y|\le 1} x^2\, dx\, dy = \frac{2}{6}$$

II način: $I=\iint_{D_3} x^2\, dx\, dy + \iint_{D_4} x^2\, dx\, dy$

$$\iint_{D_3} x^2\, dx\, dy = \int_{-1}^0 dy\int_{-y-1}^{y+1} x^2\, dx = \int_{-1}^0 dy\ \frac{1}{3}x^3\Big|_{-y-1}^{y+1} = \frac{1}{3}\int_{-1}^0 2(y+1)^3 dy = \frac{2}{3}\int_{-1}^0 (y+1)^3 dy = \frac{2}{3}\cdot\frac{1}{4}(y+1)^4\Big|_{-1}^0 = \frac{1}{6}\ \ldots$$

Izračunati dvostruki integral $\iint_S x\,dx\,dy$ gdje je područje integracije S ograničeno pravcem koji prolazi tačkama $A(2,0)$, $B(0,2)$ i lukom kruga poluprečnika 1 sa centrom u tački $C(0,1)$.

Rj.



$A(2,0)$, $B(0,2)$

krug $x^2+(y-1)^2=1$

U tačkama $A(2,0)$ i $B(0,2)$ jednačina prave glasi

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0=\frac{2-0}{0-2}(x-2)$$

$$y=-x+2$$

Nađimo presječne tačke prave i kruga

$$x^2+(y-1)^2=1$$
$$y=-x+2$$

$$x^2+(-x+2-1)^2=1$$

$$x^2+(-x+1)^2=1$$
$$x^2+x^2-2x+1=1$$
$$2x^2-2x=0$$
$$2x(x-1)=0$$

$x_1=0$, $x_2=1$

$x_1=0 \Rightarrow y=2$

$x_2=1 \Rightarrow y=1$

Presječne tačke prave i kruga su $M_1(0,2)$ i $M_2(1,1)$

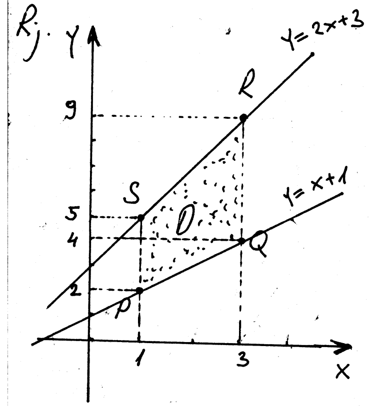
$$\iint_S x\,dx\,dy=\int_1^2\left[\int_{-y+2}^{\sqrt{1-(y-1)^2}} x\,dx\right]dy=\int_1^2 \frac12 x^2\Big|_{-y+2}^{\sqrt{1-(y-1)^2}}dy=\frac12\int_1^2\left[\left(1-(y-1)^2\right)-(2-y)^2\right]dy$$

$$=\frac12\int_1^2\left[1-y^2+2y-1-4+4y-y^2\right]dy=\frac12\int_1^2(-2y^2+6y-4)\,dy=\frac12\cdot 2\int_1^2(-y^2+3y-2)\,dy$$

$$=-\frac13 y^3\Big|_1^2+\frac32 y^2\Big|_1^2-2y\Big|_1^2=-\frac73+\frac92-2=\frac{-14+27-12}{6}=\frac16$$

Izračunati $\iint_D x\,dx\,dy$ pričemu je D četverougao $\square PQRS$ gdje su tačke $P(1,2)$, $Q(3,4)$, $R(3,9)$ i $S(1,5)$.

Rj.



$$\iint_D x\,dx\,dy=\int_1^3\left[\int_{x+1}^{2x+3} x\,dy\right]dx=$$

$y-y_1=k(x-x_1)$

$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$P(1,2)$, $Q(3,4)$

$y-2=\frac{2}{2}(x-1)$

$y=x+1$

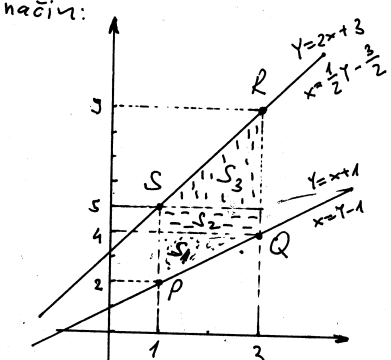
$R(3,9)$, $S(1,5)$: $y-9=\frac{-4}{-2}(x-3) \Rightarrow y=2x+3$

$$=\int_1^3 xy\Big|_{x+1}^{2x+3}dx=\int_1^3 x(2x+3-x-1)\,dx=\int_1^3(x^2+2x)\,dx=\frac13x^3\Big|_1^3+x^2\Big|_1^3=$$

$$=\frac13(27-1)+(9-1)=\frac{26}{3}+8=\frac{50}{3}$$

II način:

S: $\square PQRS$



$$\iint_S x\,dx\,dy=\iint_{S_1} x\,dx\,dy+\iint_{S_2} x\,dx\,dy+\iint_{S_3} x\,dx\,dy$$

$$\iint_{S_1} x\,dx\,dy=\int_2^4\left[\int_1^{y-1} x\,dx\right]dy=\ldots=\int_2^4\left(\frac12y^2-y\right)dy=\ldots=\frac{10}{3}$$

$$\iint_{S_2} x\,dx\,dy=\int_4^5\left[\int_1^3 x\,dx\right]dy=\ldots=\int_4^5 4\,dy=\ldots=4$$

$$\iint_{S_3} x\,dx\,dy=\int_5^9\left[\int_{\frac12y-\frac32}^3 x\,dx\right]dy=\ldots=\int_5^9\left(\frac92-\frac{(y-3)^2}{8}\right)dy=\ldots=\frac{28}{3}$$

Izračunati $\iint_D (x+y)\,dx\,dy$ ako je D oblast ograničena linijama $y^2=2x$, $x+y=4$ i $x+y=12$.

Rj. Nađimo presječne tačke ovih linija.

$$\begin{array}{l} y^2=2x \\ x+y=4 \\ \hline y^2=2x \\ y=4-x \\ \hline \end{array} \qquad \begin{array}{l} (4-x)^2=2x \\ 16-8x+x^2=2x \\ x^2-10x+16=0 \\ D=100-64=36 \\ x_{1,2}=\frac{10\pm 6}{2} \quad x_1=2,\ x_2=8 \end{array}$$

$A(2,2)$
$B(8,-4)$

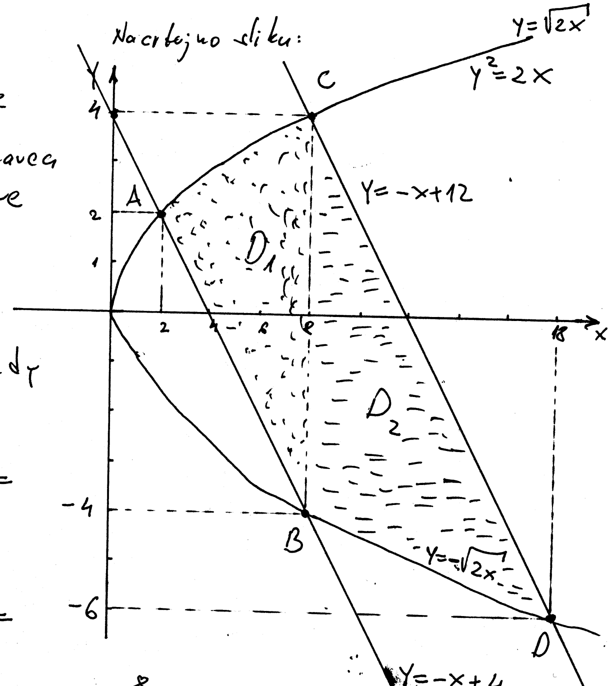
$$\begin{array}{l} y^2=2x \\ x+y=12 \\ \hline y^2=2x \\ y=12-x \\ \hline (12-x)^2=2x \\ 144-24x+x^2=2x \\ x^2-26x+144=0 \end{array} \qquad \begin{array}{l} D=676-576 \\ D=100 \\ x_{1,2}=\frac{26\pm 10}{2} \\ x_1=8 \quad x_2=18 \end{array}$$

$C(8,4)$
$D(18,-6)$

$$\begin{array}{l} x+y=4 \\ x+y=12 \\ \hline y=-x+4 \\ y=-x+12 \end{array}$$

ove dvije prave imaju isti koeficijent pravca
dvije paralelne prave

Nacrtajmo sliku:



$$I=\iint_D (x+y)\,dx\,dy=\iint_{D_1}(x+y)\,dx\,dy+\iint_{D_2}(x+y)\,dx\,dy$$

$$\iint_{D_1}(x+y)\,dx\,dy=\int_2^8\left[\int_{-x+4}^{\sqrt{2}\sqrt{x}}(x+y)\,dy\right]dx=$$

$$=\int_2^8\left(xy\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}+\frac{1}{2}y^2\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}\right)dx=$$

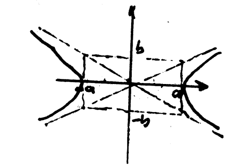
$$=\int_2^8\left[x\left(\sqrt{2}\sqrt{x}+x-4\right)+\frac{1}{2}\left(2x-(-x+4)^2\right)\right]dx=\ldots=\int_2^8\left(x+\sqrt{2}\,x^{\frac{3}{2}}+\frac{1}{2}x^2-8\right)dx=\ldots=\frac{826}{5}$$

$$\iint_{D_2}(x+y)\,dx\,dy=\int_8^{18}\left[\int_{-\sqrt{2x}}^{-x+12}(x+y)\,dy\right]dx=\ldots=\int_8^{18}\left(\sqrt{2}\,x^{\frac{3}{2}}-x-\frac{1}{2}x+72\right)dx=\ldots=\frac{5678}{15}$$

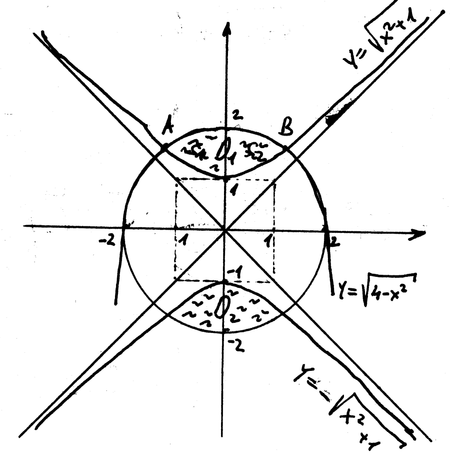
$I=543\frac{11}{15}$ vrijedn. dvostr. integr.

Izračunati $I=\iint_D dx\,dy$, ako je $D:\ y^2-x^2=1,\ x^2+y^2=4.$

Rj. Krive oblika $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo hiperbole i one su oblika



Skicirajmo naše dvije krive



$x^2+y^2=4$ je krug sa centrom u $(0,0)$ poluprečnika $r=2$

$D=D_1\cup D_2$

$$I=\iint_D dx\,dy=\iint_{D_1\cup D_2}dx\,dy=2\iint_{D_1}dx\,dy$$

$y^2=4-x^2$ $\qquad$ $y^2=x^2+1$

$y=\pm\sqrt{4-x^2}$ $\qquad$ $y=\pm\sqrt{x^2+1}$

Nađimo presječnu tačku krivih $y=\sqrt{x^2+1}$ i $y=\sqrt{4-x^2}$

$$\sqrt{x^2+1}=\sqrt{4-x^2}\quad /^2$$
$$x^2+1=4-x^2$$
$$2x^2-3=0$$
$$2x^2=3$$
$$x^2=\frac{3}{2}$$

$x_1=-\sqrt{\frac{3}{2}}$
$x_2=\sqrt{\frac{3}{2}}$

$x_1=-\sqrt{\frac{3}{2}}\Rightarrow y=\sqrt{\frac{3}{2}+1}=\sqrt{\frac{5}{2}}$

Presječne tačke su $A\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$ i $B\left(\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$.

Primjetimo da je oblast D_1 simetrična.

$$\iint_{D_1}dx\,dy=\iint_{S_1}dx\,dy+\iint_{S_2}dx\,dy=2\iint_{S_2}dx\,dy=2\int_0^{\sqrt{\frac{3}{2}}}dx\int_{\sqrt{x^2+1}}^{\sqrt{4-x^2}}dy=2\int_0^{\sqrt{\frac{3}{2}}}\left(\sqrt{4-x^2}-\sqrt{x^2+1}\right)dx$$

$$\int\sqrt{4-x^2}\,dx=2\arcsin\frac{x}{2}+\frac{x\sqrt{4-x^2}}{2}$$ (sami)

(metoda Ostrogradskog)

$$\int\sqrt{x^2+1}\,dx=\int\frac{x^2+1}{\sqrt{x^2+1}}\,dx=(ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}}\quad /'$$

$$\frac{x^2+1}{\sqrt{x^2+1}} = a\sqrt{x^2+1} + (ax+b)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} + \lambda\frac{1}{\sqrt{x^2+1}} \qquad /\cdot\sqrt{x^2+1}$$

$$x^2+1 = \underline{a}(x^2+1) + \underline{a}x^2 + \underline{\underline{b}x} + \lambda$$

$$2a = 1 \quad \Rightarrow \quad a = \frac{1}{2}$$
$$b = 0$$
$$a + \lambda = 1 \qquad \lambda = \frac{1}{2}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\int\frac{dx}{\sqrt{x^2+1}}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x+\sqrt{x^2+1}| + C$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{4-x^2}\,dx = 2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} \quad \text{(sami)}$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{x^2+1}\,dx = \frac{1}{2}\cdot\sqrt{\frac{3}{2}}\,\overbrace{\sqrt{\frac{3}{2}+1}}^{\sqrt{\frac{5}{2}}} + \frac{1}{2}\ln\left|\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{2}+1}\right| = \frac{\sqrt{15}}{4} + \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

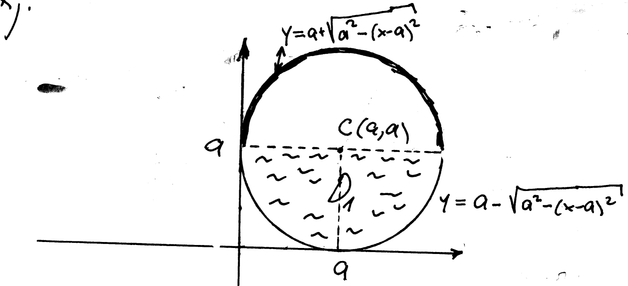
$$I = \iint_D dx\,dy = 2\iint_{D_1} dx\,dy = 2\cdot 2\iint_{S_2} dx\,dy = 4\left(2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} - \frac{\sqrt{15}}{4} - \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|\right) = 8\arcsin\frac{\sqrt{6}}{4} - 2\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

traženo rješenje.

U rješenju sljedećeg zadatka ima greška. Pronađi grešku.

(#) Izračunati dvostruki integral $\iint_S \frac{dx\,dy}{\sqrt{2a-x}}$ gdje je S krug poluprečnika a, koji dodiruje koordinatne ose i leži u prvom kvadrantu.

Rj.



krug $(x-a)^2+(y-a)^2=a^2$

$$y-a = \pm\sqrt{a^2-(x-a)^2}$$

$$y = a \pm\sqrt{a^2-(x-a)^2}$$

$$\iint_S \frac{dx\,dy}{\sqrt{2a-x}} = 2D_1$$

$$D_1 = \int_0^{2a}\left[\int_{a-\sqrt{a^2-(x-a)^2}}^{a}\frac{1}{\sqrt{2a-x}}\,dy\right]dx =$$

$$= \int_0^{2a}\frac{dx}{\sqrt{2a-x}}\cdot y\Big|_{a-\sqrt{a^2-(x-a)^2}}^{a} = \int_0^{2a}\sqrt{\frac{a^2-(x^2-2ax+a^2)}{2a-x}}\,dx = \int_0^{2a}\sqrt{\frac{a^2-x^2+2ax-a^2}{2a-x}}\,dx$$

$$= \int_0^{2a}\sqrt{\frac{x(2a-x)}{2a-x}}\,dx = \int_0^{2a}\sqrt{x}\,dx = \int_0^{2a}x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{2a} = \frac{2}{3}\cdot(2a)^{\frac{3}{2}} =$$

$$= \frac{2}{3}\sqrt{8a^3} = \frac{2}{3}\,2a\sqrt{2a} = \frac{4a}{3}\sqrt{2a}$$

Prema tome $\iint_S \frac{dx\,dy}{\sqrt{2a-x}} = \frac{8a}{3}\sqrt{2a}$

Odrediti projekcije l linije L na ravan xoy:

77. $L: 4-x^2-y^2=z, \quad z=y^2.$

78. $L: x^2+y^2=z^2, \quad (z>0), \quad x+y+z=1.$

79. $L: x^2+y^2+z^2=4, \quad x^2+y^2=z^2, \quad (z>0).$

80. $L: x^2+y^2+z^2=a^2, \quad x+y+z=0.$

81. $L: 2x+y+z=1, \quad x-y-3z=5.$

82. $L: z=x^2+y^2, \quad z=2x+2y.$

Rješenja:

77. Linija L je data kao presjek površi $z=4-x^2-y^2$ i $z=y^2$ (paraboloid i cilindar). Projekcija linije L na ravan Oxy je skup onih tačaka (x, y) za koje je aplikata z sa jedne površi jednaka aplikati z sa druge površi, dakle, taj skup određujemo iz uslova

$$4-x^2-y^2=y^2.$$

Projekcija je, dakle, elipsa

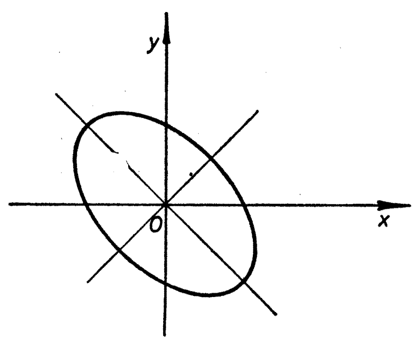
$$\frac{x^2}{2^2}+\frac{y^2}{(\sqrt{2})^2}=1.$$

78. $1=2x+2y-2xy.$

79. $x^2+y^2=2.$

80. Kriva ima projekciju

$$2x^2+2y^2+2xy-a^2=0.$$



Sl. 10

To je elipsa sa centrom u $(0, 0)$. Ose elipse su prave $y=\pm x$, a poluose su a i $\frac{a}{3}$ (sl. 10).

81. $7x+2y-8=0.$ **82.** $(x-1)^2+(y-1)^2=2.$

Po definiciji izračunati integrale:

83. $\iint\limits_{\substack{0\leqslant x\leqslant a\\0\leqslant y\leqslant b}} xydxdy.$ **84.** $\iint\limits_{\substack{0\leqslant x\leqslant 1\\0\leqslant y\leqslant 1}} x^2ydxdy.$

Rješenja:

83. Funkcija $f(x, y)=xy$ je neprekidna pa i integrabilna na pravougaoniku $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$. Podijelimo dati pravougaonik pravama $x=x_i$ $(i=1, \ldots, n)$, $y=y_j$ $(j=1, \ldots, m)$. Po definiciji je

$$\iint\limits_D xydxdy=\lim_{\substack{m\to\infty\\n\to\infty}}\sum_{i,j=1}^{n,m} f(M_{i,j})\cdot(x_i-x_{i-1})(y_j-y_{j-1})$$

pri čemu maksimalni podjeljak teži nuli kada $m\to\infty$, $n\to\infty$. Izaberimo da je:

$$x_i=\frac{a}{n}\cdot i, \quad y_j=\frac{b}{m}j,$$

i da je

$$M_{i,j}=\left(\frac{a}{n}(i-1), \quad \frac{b}{m}(j-1)\right).$$

Biće:

$$\iint\limits_{\substack{0\leqslant x\leqslant a\\0\leqslant y\leqslant b}} xydxdy=\lim_{\substack{n\to\infty\\m\to\infty}}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{a}{n}(i-1)\cdot\frac{b}{m}(j-1)\cdot\frac{ab}{n\cdot m}=$$

$$=\lim_{\substack{n\to\infty\\m\to\infty}}\frac{a^2}{n^2}\cdot\frac{b^2}{m^2}\sum_{i=1}^{n}(i-1)\sum_{j=1}^{m}(j-1)=$$

$$=\lim_{\substack{n\to\infty\\m\to\infty}}\frac{a^2b^2}{n^2\cdot m^2}\cdot\frac{(n-1)\cdot n}{2}\cdot\frac{(m-1)\cdot m}{2}=\frac{a^2b^2}{4}.$$

84. $\frac{1}{6}$.

Po definiciji izračunati integral:

$$\textbf{85.} \iint\limits_{\substack{a\leqslant x\leqslant b\\ c\leqslant y\leqslant d}} e^{x+y}\,dx\,dy.$$

Rješenja:

85. Integral postoji jer je funkcija $f(x, y)=e^{x+y}$ neprekidna. Segment $[a, b]$ podijelimo pravama $x_i=a+\dfrac{b-a}{n}\cdot i$, $(i=0, 1, \dots, n)$, a $[c, d]$ pravama $y_j=c+\dfrac{d-c}{m}\cdot j$, $(j=0, 1, \dots, m)$, i u pravougaoniku $[x_{i-1}, x_i; y_{j-1}, y_j]$ uočimo tačku $M_{ij}=\left(a+\dfrac{b-a}{n}\cdot i,\ c+\dfrac{d-c}{m}\cdot j\right)$. Formirajmo integralnu sumu

$$S_{m,n}=\sum_{i,j=1}^{n,m} f(M_{i,j})(x_i-x_{i-1})(y_i-y_{i-1})=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m} e^{a+\frac{b-a}{n}\cdot i}\cdot e^{c+\frac{d-c}{m}j}\cdot\frac{b-a}{n}\cdot\frac{d-c}{m}=$$

$$=e^a\cdot e^c\cdot\frac{b-a}{n}\cdot\frac{d-c}{m}\cdot\sum_{i=1}^{n} e^{\frac{b-a}{n}\cdot i}\cdot\sum_{j=1}^{m} e^{\frac{d-c}{m}j}.$$

Dobili smo geometrijske sume, pa je

$$S_{m,n}=e^a\cdot\frac{b-a}{n}\cdot e^{\frac{b-a}{n}}\cdot\frac{1-\left(e^{\frac{b-a}{n}}\right)^n}{1-e^{\frac{b-a}{n}}}\cdot e^c\cdot\frac{d-c}{m}\cdot e^{\frac{d-c}{m}}\cdot\frac{1-\left(e^{\frac{d-c}{m}}\right)^m}{1-e^{\frac{d-c}{m}}}=$$

$$=e^a\cdot\frac{b-a}{n}\cdot e^{\frac{b-a}{n}}\cdot\frac{1-e^{b-a}}{1-e^{\frac{b-a}{n}}}\cdot e^c\cdot\frac{d-c}{n}\cdot e^{\frac{d-c}{n}}\cdot\frac{1-e^{d-c}}{1-e^{\frac{d-c}{m}}}.$$

Kako je

$$\lim_{n\to\infty}\frac{b-a}{n}\cdot\frac{1}{1-e^{\frac{b-a}{n}}}=\lim_{\alpha\to 0}\frac{\alpha}{1-e^{\alpha}}=-1$$

i

$$\lim_{m\to\infty}\frac{d-c}{m}\,\frac{1}{1-e^{\frac{d-c}{m}}}=-1,$$

to

$$\lim_{\substack{m\to\infty\\ n\to\infty}} S_{m,n}=e^a\cdot\frac{(1-e^{b-a})}{-1}\cdot e^c\cdot\frac{(1-e^{d-c})}{-1}=$$

$$=(e^a-e^b)\cdot(e^c-e^d).$$

Dakle,

$$\iint\limits_{\substack{a\leqslant x\leqslant b\\ c\leqslant y\leqslant d}} e^{x+y}\,dx\,dy=(e^a-e^b)(e^c-e^d).$$

Promijeniti poredak integracije u sljedećim integralima uzimajući da je $f(x, y)$ neprekidna funkcija:

86. $\int_0^1 dx\int_0^x f(x, y)\,dy.$ **87.** $\int_1^e dx\int_0^{\ln x} f(x, y)\,dy.$

88. $\int_0^4 dx\int_{3x^2}^{12x} f(x, y)\,dy.$ **89.** $\int_0^1 dx\int_x^{2-x} f(x, y)\,dy.$

90. $\int_0^1 dy\int_{\sqrt{y}}^{2-y} f(x, y)\,dx.$ **91.** $\int_{-7}^{1} dx\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x, y)\,dx.$

Rješenja:

86. $\int_0^1 dy\int_y^1 f(x, y)\,dx.$ **87.** $\int_0^1 dy\int_{e^y}^{e} f(x, y)\,dx.$

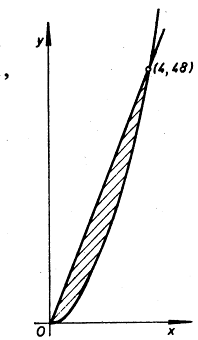
88. $x=12x\Rightarrow x=\dfrac{y}{12}$,

$y=3x^2\Rightarrow x=\sqrt{\dfrac{y}{3}}$,

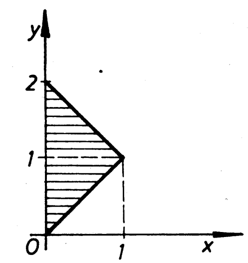
$\int_0^{48} dy\int_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x, y)\,dx$ (sl. 11).

89. $\int_0^1 dy\int_0^y f(x, y)\,dx+$

$+\int_1^2 dy\int_0^{2-y} f(x, y)\,dx.$ (sl. 12).



Sl. 11



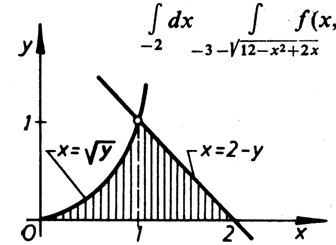
Sl. 12

90. Oblast integracije (sl. 13) dijelimo na dvije; biće

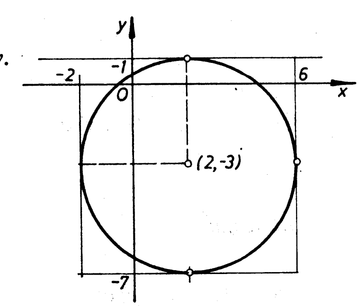
$$\int_1^2 dx\int_0^{x^2} f(x, y)\,dy+\int_1^2 dx\int_0^{2-x} f(x, y)\,dy.$$

91. Nacrtajmo linije koje ograničavaju oblast integracije $x_1=2-\sqrt{7-6y-y^2}$, $x_2=2+\sqrt{7-6y-y^2}$, $y=-7$, $y=1$. Linije x_1 i x_2 su polukružnice kružnice $(x-2)^2+(y+3)^2=16$. Oblast integracije je unutrašnjost kruga (sl. 14), pa je dati integral jednak integralu

$$\int_{-2}^{6} dx\int_{-3-\sqrt{12-x^2+2x}}^{-3+\sqrt{12-x^2+2x}} f(x, y)\,dy.$$



Sl. 13



Sl. 14

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Promijeniti poredak integracije u sljedećim integralima uzimajući da je $f(x, y)$ neprekidna funkcija:

92. $\int\limits_0^1 dy \int\limits_{2-y}^{1+\sqrt{1-y^2}} f(x, y)\, dx.$ **93.** $\int\limits_0^{\pi} dx \int\limits_0^{\sin x} f(x, y)\, dy.$

94. $\int\limits_{-1}^{0} dy \int\limits_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x, y)\, dx + \int\limits_0^8 dy \int\limits_{-2\sqrt{1+y}}^{2-y} f(x, y)\, dx.$

95. $\int\limits_0^1 dx \int\limits_{\frac{1}{2}(1-x^2)}^{\sqrt{1-x^2}} f(x, y)\, dy.$ **96.** $\int\limits_0^1 dy \int\limits_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x, y)\, dx.$

Rješenja:

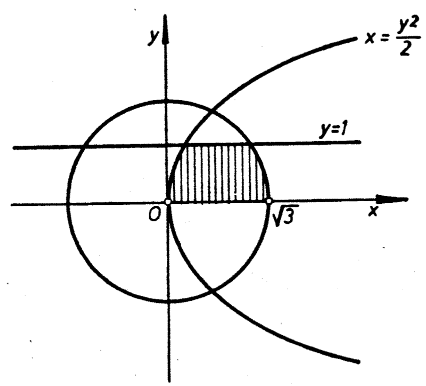
92. $\int\limits_1^2 dx \int\limits_{2-x}^{\sqrt{2x-x^2}} f(x, y)\, dy.$

93. $\int\limits_0^1 dy \int\limits_{\arcsin y}^{\pi - \arcsin y} f(x, y)\, dx.$ **94.** $\int\limits_{-6}^{2} dx \int\limits_{\frac{x^2}{4}-1}^{2-x} f(x, y)\, dy.$

95. $\int\limits_0^{1/2} dy \int\limits_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f(x, y)\, dx + \int\limits_{1/2}^{1} dy \int\limits_0^{\sqrt{1-y^2}} f(x, y)\, dx.$

96. Oblast integracije (sl. 15) ograničena je pravama $y=0$, $y=1$, lukom parabole $x=\frac{y^2}{2}$ i lukom kružnice $x^2+y^2=3$, $(x>0)$.

Prava $y=1$ i parabola sijeku se u tački sa apscisom $\frac{1}{2}$; prava $y=1$ i luk $x=\sqrt{3-y^2}$ sijeku se u tački sa apscisom $\sqrt{2}$. Otuda je



Sl. 15

$$\int\limits_0^{1/2} dx \int\limits_0^{\sqrt{2x}} f(x, y)\, dy + \int\limits_{1/2}^{1} dx \int\limits_0^{1} f(x, y)\, dy +$$

$$+ \int\limits_{\sqrt{2}}^{\sqrt{3}} dx \int\limits_0^{\sqrt{3-x^2}} f(x, y)\, dy.$$

#

Izračunati integral I:

97. $\iint\limits_{\substack{0\le x\le \frac{\pi}{2}\\ 0\le y\le \frac{\pi}{2}}} \sin(x+y)\, dxdy.$ **98.** $\iint\limits_{\substack{-1\le x\le 1\\ -2\le y\le 2}} \left(1-\frac{1}{3}x-\frac{1}{4}y\right) dxdy.$

99. $\iint\limits_{\substack{0\le x\le 1\\ 0\le y\le x}} (x^2+y^2)\, dxdy.$

Rješenja:

97. Integral ćemo izračunati svođenjem na jednostruke integrale. Biće

$$I=\int\limits_0^{\pi/2} dx \int\limits_0^{\pi/2} \sin(x+y)\, dy = \int\limits_0^{\pi/2} dx\, [-\cos(x+y)]\Big|_0^{\pi/2} = \int\limits_0^{\pi/2} \left[\cos x - \cos\left(x+\frac{\pi}{2}\right)\right] dx =$$

$$= \sin x\Big|_0^{\pi/2} - \sin\left(x+\frac{\pi}{2}\right)\Big|_0^{\pi/2} = 1+1=2.$$

98. $I=\int\limits_{-1}^{1} dx \int\limits_{-2}^{2} \left(1-\frac{1}{3}x-\frac{1}{4}y\right) dy = \int\limits_{-1}^{1} \left(y-\frac{1}{3}xy-\frac{1}{8}y^2\right)\Big|_{-2}^{2} dx =$

$$= \int\limits_{-1}^{1} \left(4-\frac{4}{3}x\right) dx = \left(4x-\frac{2}{3}x^2\right)\Big|_{-1}^{1} = 8.$$

Ako se izvrši integracija najprije po x pa po y, dobiće se isti rezultat:

$$I=\int\limits_{-2}^{2} dy \int\limits_{-1}^{1} \left(1-\frac{1}{3}x-\frac{1}{3}y\right) dx = \int\limits_{-2}^{2} \left(x-\frac{1}{6}x^2-\frac{1}{4}xy\right)\Big|_{-1}^{1} dy =$$

$$= \int\limits_{-2}^{2} \left(2-\frac{1}{2}y\right) dy = \left(2y-\frac{1}{4}y^2\right)\Big|_{-2}^{2} = 8.$$

99. $I=\int\limits_0^1 dx \int\limits_0^x (x^2+y^2)\, dy = \int\limits_0^1 dx \left(x^2 y + \frac{y^3}{3}\right)\Big|_0^x = \int\limits_0^1 \left(x^3+\frac{x^3}{3}\right) dx = \frac{1}{3}.$

Izračunati integral I:

100. $\displaystyle\iint\limits_{\substack{-3\leq x\leq 1\\ 2x-1\leq y\leq 2-x^2}} (x-y)\,dxdy\,.$

101. $\displaystyle\iint\limits_{\substack{0\leq x\leq 1\\ 0\leq y\leq 1-x}} (2x^2+y^2+1)\,dxdy.$ **102.** $\displaystyle\iint\limits_{|x|+|y|\leq 1} x^2\,dxdy.$

103. $\displaystyle\iint\limits_D (x+y)\,dxdy,$ gdje je D oblast ograničena linijama $y=x^2,\ y=x$.

Rješenja:

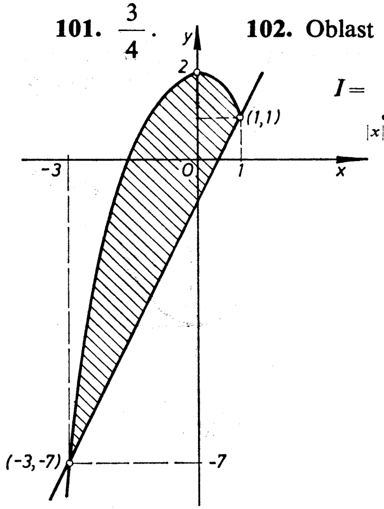
100. Prava $y=2x-1$ i parabola $y=2-x^2$ sijeku u tačkama $(-3,\ -7)$, $(1,\ 1)$ (sl. 16). Biće:

$$I=\int\limits_{-3}^{1}dx\int\limits_{2x-1}^{2-x^2}(x-y)\,dy=\int\limits_{-3}^{1}\left(xy-\frac{y^2}{2}\right)\Bigg|_{2x-1}^{2-x^2}dx=$$

$$=\int\limits_{-3}^{1}\left(2x-x^3-2+2x^2-\frac{1}{2}x^4-2x^2+x+2x^2-2x+\frac{1}{2}\right)dx=$$

$$=\int\limits_{-3}^{1}\left(-\frac{1}{2}x^4-x^3+2x^2+x-\frac{3}{2}\right)dx=4\frac{4}{15}.$$

101. $\dfrac{3}{4}$.

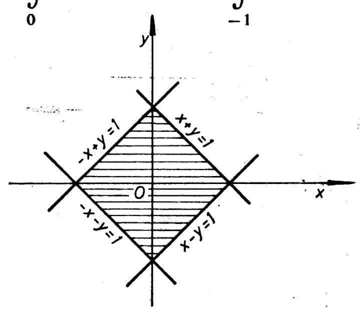


Sl. 16

102. Oblast integracije prikazana je na sl. 17. Biće:

$$I=\iint\limits_{|x|+|y|\leq 1}x^2\,dxdy=\int\limits_{0}^{1}x^2dx\int\limits_{x-1}^{1-x}dy+\int\limits_{-1}^{0}x^2dx\int\limits_{-1-x}^{1+x}dy=$$

$$=2\int\limits_{0}^{1}x^2(1-x)\,dx+2\int\limits_{-1}^{0}x^2(1+x)\,dx=\frac{1}{3}.$$



Sl. 17

103. $\displaystyle I=\int\limits_{0}^{1}dx\int\limits_{x^2}^{x}(x+y)\,dy=\int\limits_{0}^{1}\left(x^2+\frac{x^2}{2}-x^3-\frac{x^4}{2}\right)dx=\frac{3}{20}.$

Zadaci za vježbu

U zadacima 3466 — 3476 proceniti date integrale.

3466. $\iint\limits_D (x+y+10)\,d\sigma$, gde je D—krug $x^2+y^2<4$.

3467. $\iint\limits_D (x^2+4y^2+9)\,d\sigma$, gde je D—krug $x^2+y^2<4$.

3468. $\iint\limits_D (x+y+1)\,d\sigma$, gde je D—pravougaonik $0<x<1,\ 0<y<2$.

3469. $\iint\limits_D (x+xy-x^2-y^2)\,d\sigma$, gde je D—pravougaonik $0<x<1,\ 0<y<2$.

3470. $\iint\limits_D xy(x+y)\,d\sigma$, gde je D—kvadrat $0\leq x\leq 2,\ 0<y<2$.

3471. $\iint\limits_D (x+1)^y\,d\sigma$, gde je D—kvadrat $0\leq x\leq 2,\ 0<y\leq 2$.

3472. $\iint\limits_D (x^2+y^2-2\sqrt{x^2+y^2}+2)\,d\sigma$, gde je D—kvadrat $0\leq x\leq 2,\ 0\leq y\leq 2$.

3473. $\iint\limits_D (x^2+y^2-4x-4y+10)\,d\sigma$, gde je D—oblast ograničena elipsom $x^2+4y^2-2x-16y+13=0$ (uključujući granicu).

U zadacima 3477 — 3484 izračunati date dvojne integrale po pravougaonim oblastima D koje su određene nejednakostima navedenim u zadacima.

3477. $\iint\limits_D xy\,dx\,dy\quad (0<x<1,\ 0<y<2)$.

3478. $\iint\limits_D e^{x+y}\,dx\,dy\quad (0<x<1,\ 0<y<1)$.

3479. $\displaystyle\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy\quad (0<x<1,\ 0<y<1)$.

3480. $\displaystyle\iint\limits_D \frac{dx\,dy}{(x+y+1)^2}\quad (0<x<1,\ 0\leqslant y<1)$.

3481. $\displaystyle\iint\limits_D \frac{y\,dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}}\quad (0<x<1,\ 0<y<1)$.

3482. $\displaystyle\iint\limits_D x\sin(x+y)\,dx\,dy\quad \left(0<x<\pi,\ 0\leq y<\frac{\pi}{2}\right)$.

3483. $\iint\limits_D x^2ye^{xy}\,dx\,dy\quad (0\leq x\leq 1,\ 0<y<2)$.

3484. $\displaystyle\iint\limits_D x^2y\cos(xy^2)\,dx\,dy\quad \left(0<x<\frac{\pi}{2},\ 0<y<2\right)$

U zadacima 3485 — 3497 naći granice dvostrukog integrala na koji se svodi dvojni integral $\iint\limits_D f(x,\ y)\,dx\,dy$ za date (konačne) oblasti integracije D.

3485. Paralelogram koji obrazuju prave

$x=3,\quad x=5,\quad 3x-2y+4=0,\quad 3x-2y+1=0.$

3486. Trougao koji obrazuju prave $x=0,\ y=0,\ x+y=2$.

3487. $x^2+y^2\leq 1,\quad x\geq 0,\quad y\geqslant 0.$

3488. $x+y<1,\quad x-y<1,\quad x>0.$

3489. $y\geqslant x^2\quad y<4-x^2.$

3490. $\dfrac{x^2}{4}+\dfrac{y^2}{9}<1.$ **3491.** $(x-2)^2+(y-3)^2<4.$

Rješenja

3466. $8\pi(45-\sqrt{2})<I<8\pi(5+\sqrt{2})$.

3467. $36\pi<I<100\pi$.

3468. $2<I<8$. **3469.** $-8<I<\dfrac{2}{3}$.

2470. $0<I<64$.

3471. $4<I<36$.

3472. $4<I<8(5-2\sqrt{2})$. $4\pi<I<22\pi$.

3474. $0<I<\dfrac{4}{3}\pi R^5$.

3475. $24<I<72$.

3476. $29\pi\sqrt{3}<I<52\pi\sqrt{3}$.

3477. 1. **3478.** $(e-1)^2$.

3479. $\dfrac{\pi}{12}$.

3480. $\ln\dfrac{4}{3}$. **3481.** $\ln\dfrac{2+\sqrt{2}}{1+\sqrt{3}}$.

3482. $\pi-2$. **3483.** 2. **3484.** $-\dfrac{\pi}{16}$.

3485. $\displaystyle\int\limits_{3}^{5}dx\int\limits_{\frac{3x+1}{2}}^{\frac{3x+4}{2}}f(x,y)\,dy.$ **3486.** $\displaystyle\int\limits_{0}^{2}dx\int\limits_{0}^{2-x}f(x,y)\,dy.$

3487. $\displaystyle\int\limits_{0}^{1}dx\int\limits_{0}^{\sqrt{1-x^2}}f(x,y)\,dy.$ **3488.** $\displaystyle\int\limits_{0}^{1}dx\int\limits_{x-1}^{1-x}f(x,y)\,dy.$

3489. $\displaystyle\int\limits_{-\sqrt{2}}^{\sqrt{2}}dx\int\limits_{x^2}^{4-x^2}f(x,y)\,dy.$ **3490.** $\displaystyle\int\limits_{-2}^{2}dx\int\limits_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}}f(x,y)\,dy.$

3491. $\displaystyle\int\limits_{0}^{4}dx\int\limits_{3-\sqrt{4x-x^2}}^{3+\sqrt{4x-x^2}}f(x,y)\,dy.$

3492. Oblast D je ograničena parabolama $y=x^2$ i $y=\sqrt{x}$.

3493. Trougao koji obrazuju prave $y=x$, $y=2x$ i $x+y=6$.

3494. Paralelogram koji obrazuju prave

$$y=x, \quad y=x+3, \quad y=-2x+1, \quad y=-2x+5.$$

3495. $y-2x<0$, $2y-x>0$, $xy<2$.

3496. $y^2<8x$, $y\leqslant 2x$, $y+4x-24<0$.

3497. Oblast D je ograničena hiperbolom $y^2-x^2=1$ i krugom $x^2+y^2=9$ (misli se na onu oblast u kojoj leži koordinatni početak).

U zadacima 3498 — 3503 promeniti redosled integracije u datim integralima.

3498. $\int\limits_0^1 dy \int\limits_y^{\sqrt{y}} f(x, y)\,dx$. **3499.** $\int\limits_{-1}^1 dx \int\limits_0^{\sqrt{1-x^2}} f(x, y)\,dy$.

3500. $\int\limits_0^r dx \int\limits_x^{\sqrt{2rx-x^2}} f(x, y)\,dy$.

3501. $\int\limits_{-2}^2 dx \int\limits_{-\frac{1}{\sqrt{2}}\sqrt{4-x^2}}^{\frac{1}{\sqrt{2}}\sqrt{4-x^2}} f(x, y)\,dy$.

3502. $\int\limits_1^2 dx \int\limits_x^{2x} f(x, y)\,dy$. **3503.** $\int\limits_0^2 dx \int\limits_{2x}^{6-x} f(x, y)\,dy$.

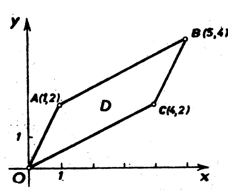
3504. Izmenivši redosled integracije predstaviti dati izraz u obliku jednog dvostrukog integrala:

1) $\int\limits_0^1 dx \int\limits_0^x f(x, y)\,dy+\int\limits_1^2 dx \int\limits_0^{2-x} f(x, y)\,dy$;

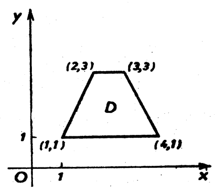
2) $\int\limits_0^1 dx \int\limits_0^{x^2} f(x, y)\,dy+\int\limits_1^3 dx \int\limits_0^{\frac{1}{2}(3-x)} f(x, y)\,dy$;

3) $\int\limits_0^1 dx \int\limits_0^{x^{\frac{2}{3}}} f(x, y)\,dy+\int\limits_1^2 dx \int\limits_0^{1-\sqrt{4x-x^2-3}} f(x, y)\,dy$.

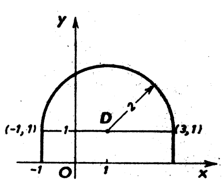
3505. Predstaviti dvojni integral $\iint\limits_D f(x, y)\,dx\,dy$ po oblastima D prikazanim na sl. 62, 63, 64, 65, — u obliku zbira dvostrukih integrala (sa najmanjim brojem sabiraka). Granične linije oblasti prikazanih na sl. 64 i 65 sastoje se iz pravolinijskih odsečaka i kružnih lukova.



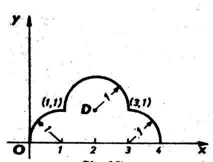
Sl. 62



Sl. 63



Sl. 64



Sl. 65

Rješenja

3492. $\int\limits_0^1 dx \int\limits_{x^2}^{\sqrt{x}} f(x, y)\,dy$.

3493. $\int\limits_0^2 dx \int\limits_x^{2x} f(x, y)\,dy+\int\limits_2^3 dx \int\limits_x^{6-x} f(x, y)\,dy$.

3494. $\int\limits_{-\frac{2}{3}}^{\frac{1}{3}} dx \int\limits_{1-2x}^{x+3} f(x, y)\,dy+\int\limits_{\frac{1}{3}}^{\frac{2}{3}} dx \int\limits_x^{x+3} f(x, y)\,dy+\int\limits_{\frac{2}{3}}^{\frac{5}{3}} dx \int\limits_x^{5-2x} f(x, y)\,dy$.

3495. $\int\limits_0^1 dx \int\limits_{\frac{x}{2}}^{2x} f(x, y)\,dy+\int\limits_1^2 dx \int\limits_{\frac{x}{2}}^{\frac{2}{x}} f(x, y)\,dy$.

3496. $\int\limits_0^2 dx \int\limits_{-2\sqrt{2x}}^{2x} f(x, y)\,dy+\int\limits_2^{\frac{9}{2}} dx \int\limits_{-2\sqrt{2x}}^{2\sqrt{2x}} f(x, y)\,dy+\int\limits_{\frac{9}{2}}^8 dx \int\limits_{-2\sqrt{2x}}^{24-4x} f(x, y)\,dy$.

3497. $\int\limits_{-3}^{-2} dx \int\limits_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dx+\int\limits_{-2}^2 dx \int\limits_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\,dy+\int\limits_2^3 dx \int\limits_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\,dy$.

3498. $\int\limits_0^1 dx \int\limits_{x^2}^x f(x, y)\,dy$. **3499.** $\int\limits_0^1 dy \int\limits_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\,dx$.

3500. $\int\limits_0^r dy \int\limits_{r-\sqrt{r^2-y^2}}^{y} f(x, y)\,dx$. **3501.** $\int\limits_{-\sqrt{2}}^{\sqrt{2}} dy \int\limits_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} f(x, y)\,dx$.

3502. $\int\limits_1^2 dy \int\limits_1^y f(x, y)\,dx+\int\limits_2^4 dy \int\limits_{\frac{y}{2}}^2 f(x, y)\,dx$.

3503. $\int\limits_0^4 dy \int\limits_0^{\frac{y}{2}} f(x, y)\,dx+\int\limits_4^6 dy \int\limits_0^{6-y} f(x, y)\,dx$.

3504. 1) $\int\limits_0^1 dy \int\limits_y^{2-y} f(x, y)\,dx$; 2) $\int\limits_0^1 dy \int\limits_{\sqrt{y}}^{3-2y} f(x, y)\,dx$;

3) $\int\limits_0^1 dy \int\limits_{y^{\frac{3}{2}}}^{2-\sqrt{2y-y^2}} f(x, y)\,dx$.

3505. 1) $\int\limits_0^2 dy \int\limits_{\frac{y}{2}}^{2y} f(x, y)\,dx+\int\limits_2^4 dy \int\limits_{2y-3}^{\frac{y+6}{2}} f(x, y)\,dx$;

2) $\int\limits_1^3 dy \int\limits_{\frac{y+1}{2}}^{\frac{9-y}{2}} f(x, y)\,dx$; 3) $\int\limits_{-1}^3 dx \int\limits_0^{1+\sqrt{3+2x-x^2}} f(x, y)\,dy$;

4) $\int\limits_0^1 dy \int\limits_{1-\sqrt{1-y^2}}^{3+\sqrt{1-y^2}} f(x, y)\,dx+\int\limits_1^2 dy \int\limits_{2-\sqrt{2y-y^2}}^{2+\sqrt{2y-y^2}} f(x, y)\,dx$.

U zadacima 3506 — 3512 izračunati date integrale:

3506. 1) $\int\limits_0^a dx \int\limits_0^{\sqrt{x}} dy$; 2) $\int\limits_2^4 dx \int\limits_x^{2x} \frac{y}{x}\,dy$; 3) $\int\limits_1^2 dy \int\limits_0^{\ln y} e^x\,dx$.

3507. $\iint\limits_D x^3 y^2\,dx\,dy$, D—**krug** $x^2+y^2<R^2$.

3508. $\iint\limits_D (x^2+y)\,dx\,dy$, **oblast** D **je ograničena parabolama** $y=x^2$ **i** $y^2=x$.

3509. $\iint\limits_D \frac{x^2}{y^2}\,dx\,dy$, **oblast** D **je ograničena pravama** $x=2$, $y=x$ **i hiperbolom** $xy=1$.

3510. $\iint\limits_D \cos(x+y)\,dx\,dy$, **oblast** D **je ograničena pravama** $x=0$, $y=\pi$ **i** $y=x$.

3511. $\iint\limits_D \sqrt{1-x^2-y^2}$ **oblast** D **je četvrtina kruga** $x^2+y^2<1$, **koji leži u prvom kvadrantu.**

3512. $\iint\limits_D x^2 y^2 \sqrt{1-x^3-y^3}\,dx\,dy$, **oblast** D **je ograničena krivom** $x^3+y^3=1$ **i koordinatnim osama.**

3513. Naći srednju vrednost funkcije $z=12-2x-3y$ **u oblasti ograničenoj pravama** $12-2x-3y=0$, $x=0$, $y=0$.

3514. Naći srednju vrednost funkcije $z=2x+y$ **u oblasi ograničenoj pravom** $x+y=3$ **i koordinatnim osama.**

3515. Naći srednju vrednost funkcije $z=x+6y$ **u oblasti ograničenoj pravama** $y=x$, $y=5x$ **i** $x=1$.

3516. Naći srednju vrednost funkcije $z=\sqrt{R^2-x^2-y^2}$ **u krugu** $x^2+y^2<R^2$.

Rješenja

3506. 1) $\frac{2}{3}a^{\frac{3}{2}}$; 2) 9; 3) $\frac{1}{2}$. **3507.** 0. **3508.** $\frac{33}{140}$. **3509.** $\frac{9}{4}$.

3510. -2. **3511.** $\frac{\pi}{6}$. **3512.** $\frac{4}{135}$. **3513.** 4. **3514.** 3. **3515.** $12\frac{2}{3}$.

3516. $\frac{2}{3}R$.

Smjena promjenjivih u dvostrukom integralu

Neka je dat integral $I=\iint\limits_D f(x,y)\,dx\,dy$.

Ako uvodimo nove promjenjive u i v takve da je
$x=\varphi(u,v)$
$y=\psi(u,v)$ zada se oblast D preslikava u D'. Jakobijan transformacijom $J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$ i imamo

$$I=\iint\limits_{D'} f(\varphi(u,v),\ \psi(u,v))\,|J|\,du\,dv,$$

$$dx\,dy=|J|\,du\,dv$$

Npr. smjena polarnim koordinatama izgleda
$x=r\cos\varphi$
$y=r\sin\varphi$, r i φ su polarne koordinate, $r\geq 0$
$0\leq\varphi\leq 2\pi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r(\sin^2\varphi+\cos^2\varphi)=r$$

$$\iint\limits_D f(x,y)\,dx\,dy=\iint\limits_{D'} f(r\cos\varphi, r\sin\varphi)\cdot|r|\,d\varphi\,dr$$

Polarne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u jednačinama koje opisuju oblast integracije pojavljuje izraz x^2+y^2.

Poopštene polarne koordinate izgledaju $x=ar\cos\varphi$ $(a>0)$
$y=br\sin\varphi$ $(b>0)$

$J=\ldots=abr$ (ostavljamo za vježbu kako doći do ovog rezultata)

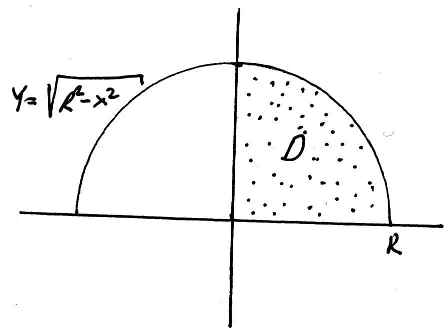
\# Dati dvostruki integral $\int\limits_0^R dx\int\limits_0^{\sqrt{R^2-x^2}} f(x,y)\,dy$

iz pravouglaonih koordinata transformisati na polarne koordinate.

Rj. Oblast integracije D prema postavci zadatka je

$$D:\begin{cases}0\leq x\leq R\\ 0\leq y\leq\sqrt{R^2-x^2}\end{cases}$$

Skicirajmo oblast D.



$y^2=R^2-x^2$

$x^2+y^2=R^2$

Polarne koordinate glase
$x=r\cos\varphi$
$y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

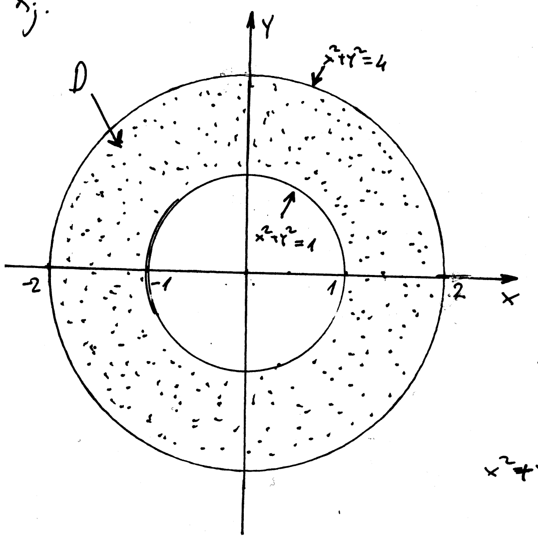
$$D\xrightarrow{\text{transformise}}D':\begin{cases}0\leq r\leq R\\ 0\leq\varphi\leq\frac{\pi}{2}\end{cases}$$

$$\int\limits_0^R dx\int\limits_0^{\sqrt{R^2-x^2}} f(x,y)\,dy=\int\limits_0^R r\,dr\int\limits_0^{\pi/2} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$

Izračunati dvostruki integral $I=\iint\limits_D \frac{dx\,dy}{\sqrt{x^2+y^2}}$ gdje je

D - kružni kolut, oblast omeđen krugovima $x^2+y^2=1$ i $x^2+y^2=4$ (drugim riječima $D=\{(x,y) \mid x,y\in\mathbb{R}\ ;\ 1\le x^2+y^2\le 4\}$.

Rj.



Zadatak ćemo riješiti prelaskom na polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$D \xrightarrow{\text{transformiše}} D'$

gdje je

$$D'=\begin{cases}1\le r\le 2\\ 0\le\varphi\le 2\pi\end{cases}$$

$$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$$

$$\iint\limits_D \frac{dx\,dy}{\sqrt{x^2+y^2}}=\iint\limits_{D'}\frac{r\,dr\,d\varphi}{\sqrt{r^2}}=\iint\limits_{D'} dr\,d\varphi=\int_1^2 dr\int_0^{2\pi} d\varphi=2\pi\cdot 1=2\pi$$

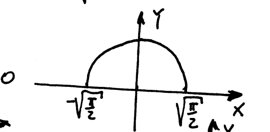
Izračunati dvostruki integral

$$I=\int_0^{\sqrt{\frac{\pi}{2}}} dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy\,.$$

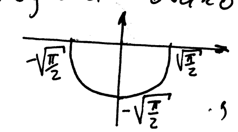
Rj. Oblast integracije D je

$$D:\begin{cases}0\le x\le\sqrt{\frac{\pi}{2}}\\ -\sqrt{\frac{\pi}{2}-x^2}\le y\le\sqrt{\frac{\pi}{2}-x^2}\end{cases}$$

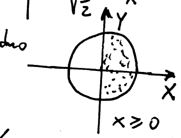
Znamo da f-ja $y=\sqrt{\frac{\pi}{2}-x^2}$ izgleda ovako



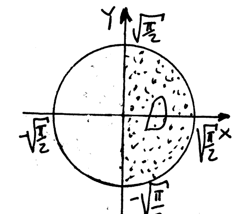
dok f-ja $y=-\sqrt{\frac{\pi}{2}-x^2}$ izgleda



, sve zajedno $\Rightarrow$



Ove dvije f-je se dobiju iz f-je $y^2=\frac{\pi}{2}-x^2$ tj. $x^2+y^2=\frac{\pi}{2}$ što predstavlja jednačinu kruga sa centrom u koordinatnom početku, poluprečnika $\sqrt{\frac{\pi}{2}}$.



Uvedimo polarne koordinate

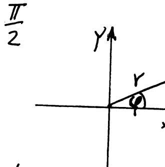
$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

$D \xrightarrow{\text{transform.}} D':\begin{cases}0\le r\le\sqrt{\frac{\pi}{2}}\\ -\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}\end{cases}$

$x^2+y^2=\dots=r^2$



$$I=\int_0^{\sqrt{\frac{\pi}{2}}}dx\int_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}}\cos(x^2+y^2)\,dy=\iint\limits_D\cos(x^2+y^2)\,dx\,dy=\iint\limits_{D'}\cos(r^2)\,r\,dr\,d\varphi=$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi\int_0^{\sqrt{\frac{\pi}{2}}}r\cos(r^2)\,dr=\left|\begin{array}{l}r^2=t\\ 2r\,dr=dt\\ r\,dr=\frac{1}{2}dt\\ r\Big|_0^{\sqrt{\frac{\pi}{2}}}\Rightarrow t\Big|_0^{\frac{\pi}{2}}\end{array}\right|=\frac{1}{2}\int_{-\pi/2}^{\pi/2}d\varphi\int_0^{\pi/2}\cos t\,dt=\frac{1}{2}\cdot\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\sin t\Big|_0^{\frac{\pi}{2}}$$

$$=\frac{1}{2}\cdot\pi\cdot 1=\frac{\pi}{2}$$ traženo rješenje

Izračunati $\iint\limits_D (x^2+y^2)\,dx\,dy$ gdje je D unutrašnjost kruga $x^2+y^2=2ax$.

Rj. $x^2+y^2=2ax$

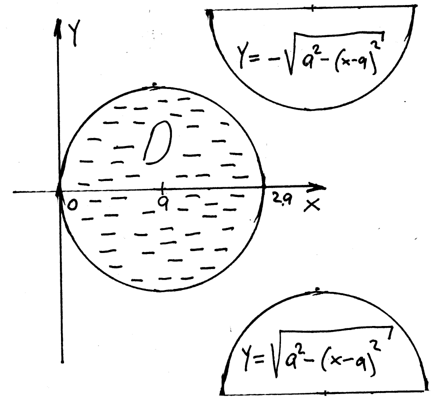
$x^2-2ax+y^2=0$

$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

$(x-a)^2+y^2=a^2$

$S(a,0)$ centar

poluprečnik a

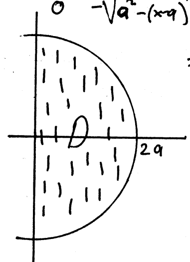


$$\iint\limits_D (x^2+y^2)\,dx\,dy=\int_0^{2a}\left(\int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}}(x^2+y^2)\,dy\right)dx=$$

$x=r\cos\varphi$

$y=r\sin\varphi$

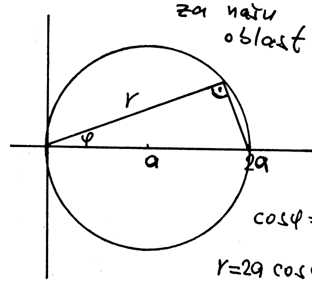
$x^2+y^2=r^2$



za ovakvu oblast imali bi

$0\le r\le 2a$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$



za našu oblast

$\cos\varphi=\frac{r}{2a}$

$r=2a\cos\varphi$

$0\le r\le 2a\cos\varphi$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$dx\,dy=|J|\,dr\,d\varphi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$=$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2a\cos\varphi}r^2|r|\,dr\right]d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2a\cos\varphi}r^3\,dr\right]d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}r^4\Big|_0^{2a\cos\varphi}d\varphi=4a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,d\varphi$$

$$=\left|\cos^4\varphi=(\cos^2\varphi)^2=\left(\frac{1+\cos 2\varphi}{2}\right)^2=\frac{1}{4}\left(\cos^2 2\varphi+2\cos 2\varphi+1\right)\right|=$$

$1=\sin^2\varphi+\cos^2\varphi$

$\cos 2\varphi=\cos^2\varphi-\sin^2\varphi$

$$=a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2 2\varphi+2\cos 2\varphi+1)\,d\varphi=a^4\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}(1+\cos 4\varphi)\,d\varphi+2\cdot\frac{1}{2}\sin 2\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right]\overset{(*)}{=}$$

$1=\sin^2 2\varphi+\cos^2 2\varphi$

$\cos 4\varphi=\cos^2 2\varphi-\sin^2 \varphi^2$

$1+\cos 4\varphi=2\cos^2 2\varphi$

$\cos^2 2\varphi=\frac{1}{2}(1+\cos 4\varphi)$

$$\int\cos 2\varphi\,d\varphi=\left|\begin{array}{l}2\varphi=t\\ 2d\varphi=dt\\ d\varphi=\frac{1}{2}dt\end{array}\right|=\frac{1}{2}\int\cos t\,dt=\frac{1}{2}\sin t+c=\frac{1}{2}\sin 2\varphi+c$$

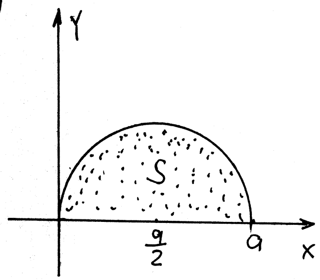
$$\overset{(*)}{=}a^4\left[\frac{1}{2}\pi+\frac{1}{2}\cdot\frac{1}{4}\sin 4\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+0+\pi\right]=a^4\left[\frac{3\pi}{2}+\frac{1}{8}\cdot 0\right]=\frac{3\pi}{2}a^4$$

II način: Uvodimo smjenu $x=a+r\cos\varphi$, $y=r\sin\varphi$, $0\le\varphi\le 2\pi$, $0\le r\le a$

URADITI ZA VJEŽBU

Izračunati integral $\iint_S y\,dx\,dy$ gdje je S unutrašnjost gornjeg polukruga poluprečnika $\frac{a}{2}$ sa središtem u tački $\left(\frac{a}{2},0\right)$.

Rj.



I način:

$$\iint_S y\,dx\,dy = \ldots = \int_0^{\frac{\pi}{2}}\left[\int_0^{a\cos\varphi} r\sin\varphi\cdot|r|\,dr\right]d\varphi$$

$$= \ldots = \frac{a^3}{12}$$

OSTAVJAMO ZA VJEŽBU KAKO SMO OVO DOBILI

II način:

$$\iint_S y\,dx\,dy = \left|\begin{matrix} x=\frac{a}{2}+r\cos\varphi \\ y=r\sin\varphi \\ 0\le\varphi\le\pi \\ 0\le r\le\frac{a}{2}\end{matrix}\right|$$

$dx\,dy=|J|\,dr\,d\varphi$

$$J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi\end{vmatrix}$$

$J=r$

$$=\int_0^{\pi}\left[\int_0^{\frac{a}{2}} r\sin\varphi\cdot|r|\,dr\right]d\varphi = \int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}\,d\varphi = \frac{a^3}{24}\int_0^{\pi}\sin\varphi\,d\varphi =$$

$$=\frac{a^3}{24}(-\cos\varphi)\Big|_0^{\pi} = -\frac{a^3}{24}(-1-1)=\frac{a^3}{12}$$

Izračunati dvostruki integral dat u polarnim koordinatama $I=\iint_D \rho\sin\varphi\,d\rho\,d\varphi$ gdje je oblast D

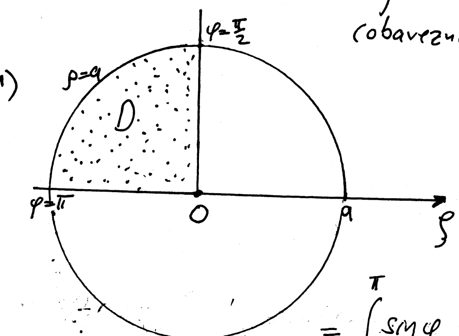
a) kružni sektor, ograničen linijama $\rho=a$, $\varphi=\frac{\pi}{2}$ i $\varphi=\pi$

b) polukrug $\rho\le 2a\cos\varphi$, $0\le\varphi\le\frac{\pi}{2}$

c) oblast između linija $\rho=2+\cos\varphi$ i $\rho=1$.

(obavezno nacrtati izgled oblasti D)

Rj. a)

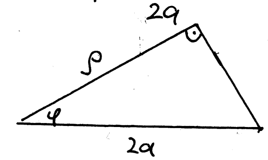


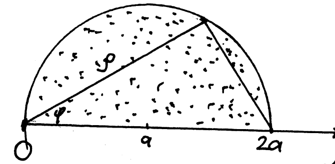
$$I=\iint_D \rho\sin\varphi\,d\rho\,d\varphi = \int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,d\varphi\int_0^a\rho\,d\rho =$$

$$=\int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,\frac{\rho^2}{2}\Big|_0^a\,d\varphi = \frac{a^2}{2}\left(-\cos\varphi\Big|_{\frac{\pi}{2}}^{\pi}\right)=\frac{a^2}{2}$$

b) $\rho=2a\cos\varphi$

$\cos\varphi=\frac{\rho}{2a}$





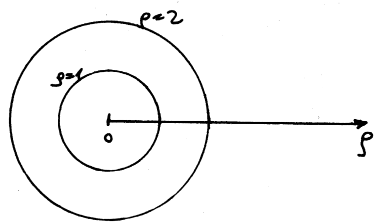
$$D:\begin{cases}0\le\rho\le 2a\cos\varphi \\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$$

$$I=\iint_D\rho\sin\varphi\,d\rho\,d\varphi = \int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho = \int_0^{\frac{\pi}{2}}\frac{1}{2}\rho^2\Big|_0^{2a\cos\varphi}\sin\varphi\,d\varphi =$$

$$=\frac{1}{2}\cdot 4a^2\int_0^{\frac{\pi}{2}}\sin\varphi\cos^2\varphi\,d\varphi = \left|\begin{matrix}\cos\varphi=t \\ -\sin\varphi\,d\varphi=dt \\ \varphi\Big|_0^{\frac{\pi}{2}}\Rightarrow t\Big|_1^0\end{matrix}\right| = 2a^2\left(-\int_1^0 t^2\,dt\right)=$$

$$=2a^2\int_0^1 t^2\,dt = 2a^2\cdot\frac{t^3}{3}\Big|_0^1 = \frac{2}{3}a^2$$ traženo rješenje

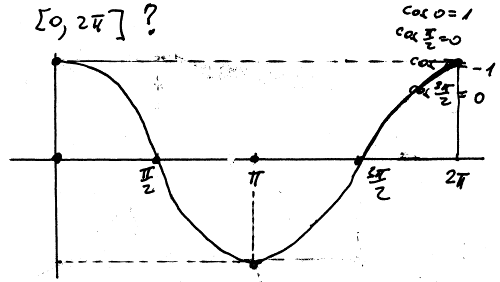
c) linije $\rho=1$ i $\rho=2$ nije teško nacrtati



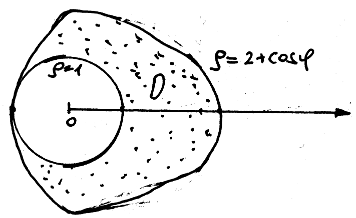
Problem predstavlja linija

$\rho = 2+\cos\varphi$

Kako izgleda $\cos\varphi$ na intervalu $[0, 2\pi]$?



Ako liniji $\rho=2$ dodamo $\cos\varphi$ imamo otprilike sljedeću sliku:



$$D: \begin{cases} 1 \le \rho \le 2+\cos\varphi \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$I=\iint_D \rho\sin\varphi \, d\rho\, d\varphi = \int_0^{2\pi} \sin\varphi\, d\varphi \int_1^{2+\cos\varphi} \rho\, d\rho = \int_0^{2\pi} \frac{\rho^2}{2}\Big|_1^{2+\cos\varphi} \sin\varphi\, d\varphi =$$

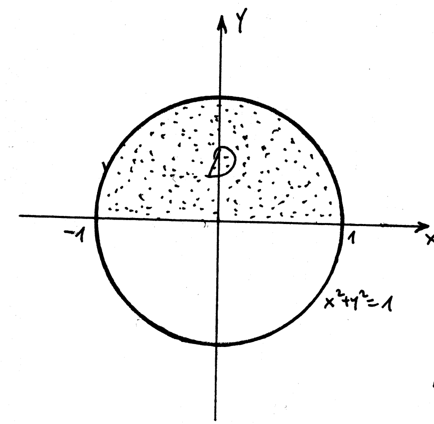
$$= \frac{1}{2}\int_0^{2\pi} \left((2+\cos\varphi)^2 - 1^2\right)\sin\varphi\, d\varphi = \frac{1}{2}\int_0^{2\pi} (4+4\cos\varphi+\cos^2\varphi-1)\sin\varphi\, d\varphi$$

$$= -\frac{1}{2}\int_0^{2\pi} (\cos^2\varphi+4\cos\varphi+3)\, d\cos\varphi = \left(-\frac{1}{2}\right)\left(\frac{\cos^3\varphi}{3}\Big|_0^{2\pi} + 4\,\frac{\cos^2\varphi}{2}\Big|_0^{2\pi} + 3\cos\varphi\Big|_0^{2\pi}\right)$$

$$= \left(-\frac{1}{2}\right)\left(\frac{1}{3}(1-1) + 2(1-1) + 3(1-1)\right) = 0$$ traženo rješenje

Izračunati integral $I=\iint_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, ako je D oblast data sa: $x^2+y^2 \le 1$, $y \ge 0$.

Rj. Skicirajmo oblast D



$$D: \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$$

Uvedimo polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$

$$D \xrightarrow{\text{transformacije}} D', \quad D': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \end{cases}$$

$1-x^2-y^2 = 1-(x^2+y^2) = 1-r^2$

$1+x^2+y^2 = 1+r^2$

$$I=\iint_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy = \iint_{D'} \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr\,d\varphi = \int_0^{\pi} d\varphi \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr$$

Izračunajmo posebno drugi integral

$$\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr = \int_0^1 \frac{\sqrt{1-r^2}}{\sqrt{1+r^2}}\, r\,dr = \int_0^1 \frac{1-r^2}{\sqrt{(1+r^2)(1-r^2)}}\cdot r\,dr = \int_0^1 \frac{r}{\sqrt{1-r^4}}\,dr - \int_0^1 \frac{r^3}{\sqrt{1-r^4}}\,dr$$

$r|_0^1 \Rightarrow t|_0^1$

$$\int_0^1 \frac{r}{\sqrt{1-r^4}}\,dr = \left| \begin{matrix} r^2=t \\ 2r\,dr=dt \\ r\,dr=\frac{1}{2}dt \end{matrix} \right| = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\arcsin t\Big|_0^1 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

$$\int_0^1 \frac{r^3}{\sqrt{1-r^4}}\,dr = \left| \begin{matrix} 1-r^4=s^2 & r^3dr=-\frac{1}{2}s\,ds \\ -4r^3dr=2s\,ds & r|_0^1 \Rightarrow s|_1^0 \end{matrix} \right| = -\frac{1}{2}\int_1^0 \frac{s\,ds}{\sqrt{s^2}} = \frac{1}{2}$$

$$I=\int_0^{\pi} d\varphi \int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,dr = \varphi\Big|_0^{\pi}\cdot\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi^2}{4}-\frac{\pi}{2}$$ traženo rješenje

Izračunati integral $I=\int\limits_0^{\sqrt{3}/2}dx\int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{x^2+y^2}\,dy$.

Rj. Pokušajmo prvo skicirati oblast integracije D. Primijetimo da se u drugom integralu pojavljuju f-je $y=\sqrt{1-x^2}$ i $y=1-\sqrt{1-x^2}$. Nacrtajmo ih.

$y=\sqrt{1-x^2}$

$y^2=1-x^2$

$x^2+y^2=1$

krug sa centrom u $C(0,0)$ poluprečnika $r=1$

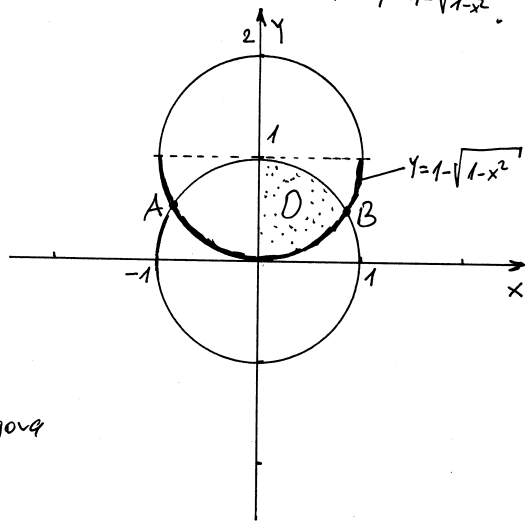
$y=1-\sqrt{1-x^2}$

$y-1=-\sqrt{1-x^2}$

$(y-1)^2=1-x^2$

$x^2+(y-1)^2=1$

krug sa centrom u $C(0,1)$ poluprečnika $r=1$



Pronađimo tačke presjeka ovih krugova

$$\frac{\begin{array}{l}x^2+y^2=1\\ x^2+(y-1)^2=1\end{array}}{\begin{array}{l}x^2=1-y^2\\ x^2+(y-1)^2=1\end{array}}$$

$1-y^2+(y-1)^2=1$

$\cancel{1}-\cancel{y^2}+\cancel{y^2}-2y+1=\cancel{1}$

$1-2y=0$

$2y=1$

$y=\frac{1}{2}$

$x^2+y^2=1$

$x^2=1-y^2$

$x^2=1-\frac{1}{4}$

$x^2=\frac{3}{4}$

$x_{1,2}=\pm\frac{\sqrt{3}}{2}$

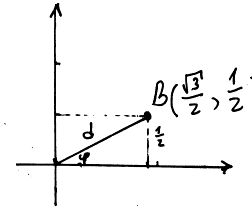
Tačke presjeka su $A(-\frac{\sqrt{3}}{2},\frac{1}{2})$ i $B(\frac{\sqrt{3}}{2},\frac{1}{2})$

Sad možemo konačno nacrtati oblast integracije D.

$$I=\int\limits_0^{\sqrt{3}/2}dx\int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{x^2+y^2}\,dy=\iint\limits_D\sqrt{x^2+y^2}\,dx\,dy$$

Oblast D ćemo podijeliti na dva dijela D_1 i D_2 pa ćemo imati

$$\iint\limits_D\sqrt{x^2+y^2}\,dx\,dy=\iint\limits_{D_1}\sqrt{x^2+y^2}\,dx\,dy+\iint\limits_{D_2}\sqrt{x^2+y^2}\,dx\,dy.$$

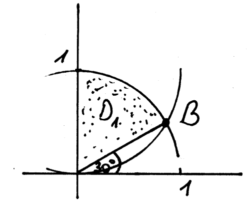


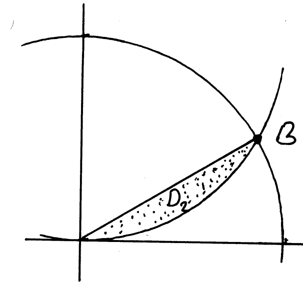
$d=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$\sin\varphi=\frac{\frac{1}{2}}{1}$

$\sin\varphi=\frac{1}{2}$

$\varphi=30^\circ$

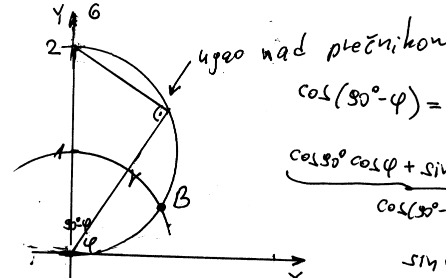




$$\iint\limits_{D_1}\sqrt{x^2+y^2}\,dx\,dy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi\end{array}\right.$$

$$D_1\xrightarrow{\text{transformiše}}D_1':\begin{cases}0\le r\le 1\\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\end{cases}\Bigg|=\iint\limits_{D_1'}\sqrt{r^2}\,r\,dr\,d\varphi=\int\limits_0^1 r^2dr\int\limits_{\pi/6}^{\pi/2}d\varphi=$$

$$=\varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}\cdot\frac{1}{3}r^3\Big|_0^1=\frac{1}{3}\left(\frac{3\pi}{6}-\frac{\pi}{6}\right)=\frac{1}{3}\cdot\frac{2\pi}{6}=\frac{1}{3}\cdot\frac{\pi}{3}=\frac{\pi}{9}$$



$\cos(90^\circ-\varphi)=\frac{r}{2}$

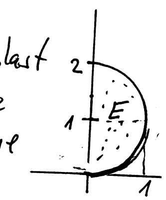
$\underbrace{\cos 90^\circ\cos\varphi+\sin 90^\circ\sin\varphi}_{\cos(90^\circ-\varphi)}=\frac{r}{2}$

$\sin\varphi=\frac{r}{2}$

$r=2\sin\varphi$

Prema tome pomoćna oblast E ima granice

$E:\begin{cases}0\le r\le 2\sin\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$



Odavde možemo vidjeti polarne granice za D_2

$$\iint\limits_{D_2}\sqrt{x^2+y^2}\,dx\,dy=\left|\begin{array}{l}\text{uvedimo polarne koordinate}\\ x=r\cos\varphi\\ y=r\sin\varphi\\ dx\,dy=r\,dr\,d\varphi\end{array}\right.\quad D_2\xrightarrow{\text{transformiše}}D_2':\begin{cases}0\le r\le 2\sin\varphi\\ 0\le\varphi\le\frac{\pi}{6}\end{cases}\Bigg|$$

$$=\iint\limits_{D_2'}r^2\,dr\,d\varphi=\int\limits_0^{\pi/6}d\varphi\int\limits_0^{2\sin\varphi}r^2dr=\int\limits_0^{\pi/6}\frac{1}{3}r^3\Big|_0^{2\sin\varphi}d\varphi=\frac{8}{3}\int\limits_0^{\pi/6}\sin^3\varphi\,d\varphi\overset{\text{za vježbu}}{=}\dots=-\sqrt{3}+\frac{16}{9}$$

Prema tome $I=\frac{\pi}{9}+\frac{16}{9}-\sqrt{3}=\frac{\pi+16}{9}-\sqrt{3}$ traženo rješenje

(#) Dati dvostruki integral $\int\limits_{\frac{R}{2}}^{2R} dy \int\limits_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx$

iz pravouganih koordinata transformisati na polarne koordinate.

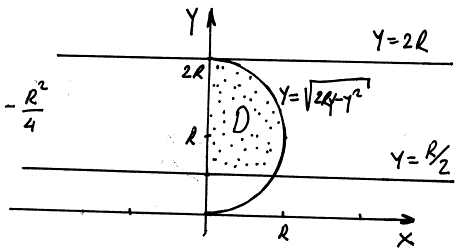
Rj. Skicirajmo oblast integracije.

Iz postavke vidimo da je x ograničen sa pravom $x=0$ i krivom $x=\sqrt{2Ry-y^2}$

$$D:\begin{cases} 0 \le x \le \sqrt{2Ry-y^2} \\ 2R \le y \le R/2 \end{cases}$$

$$x^2 = 2Ry - y^2$$
$$x^2 + y^2 - 2\cdot y\cdot R + R^2 - R^2 = 0$$
$$x^2 + (y-R)^2 = R^2$$

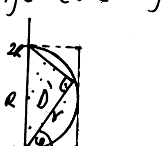
krug sa centrom u tački $(0,R)$ poluprečnika R.



Polarne koordinate glase

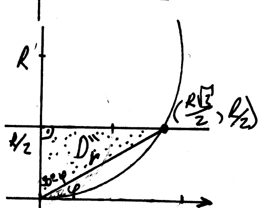
$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$dx\,dy = r\,dr\,d\varphi$$

Da bi došli do ideje kako opisati oblast D posmatrajmo sljedeće "jednostavnije" oblasti D' i D'':



$$\cos(90^\circ-\varphi) = \frac{r}{2R} \Rightarrow r = 2R\sin\varphi$$
$$\cos(90^\circ-\varphi) = \sin\varphi$$

$$D':\begin{cases} 0 \le r \le 2R\sin\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$



$$\cos(90^\circ-\varphi) = \frac{R/2}{r}$$
$$\sin\varphi = \frac{R}{2r}$$
$$2r = \frac{R}{\sin\varphi} \Rightarrow r = \frac{R}{2\sin\varphi}$$

$$D'':\begin{cases} 0 \le r \le \frac{R}{2\sin\varphi} \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \end{cases}$$

Sad nije teško vidjeti da će oblast D opisana pomoću polarnih koordinata postati

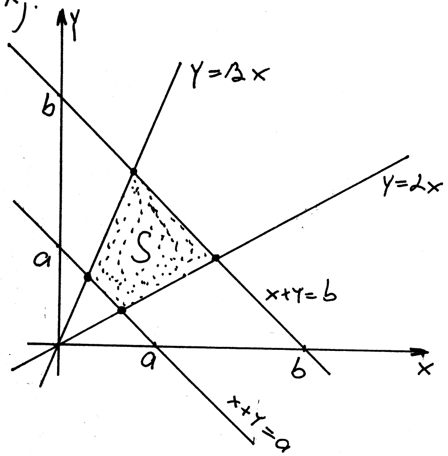
$$D:\begin{cases} \frac{R}{2\sin\varphi} \le r \le 2R\sin\varphi \\ \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \end{cases}.$$

Prema tome

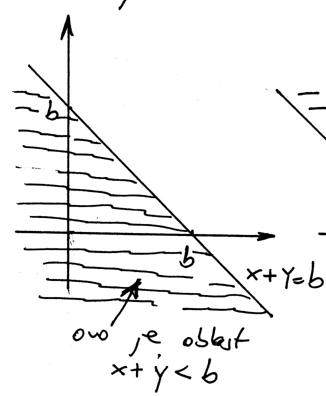
$$\int\limits_{R/2}^{2R} dy \int\limits_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx = \int\limits_{\pi/6}^{\pi/2} d\varphi \int\limits_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(r\cos\varphi, r\sin\varphi)\,r\,dr$$

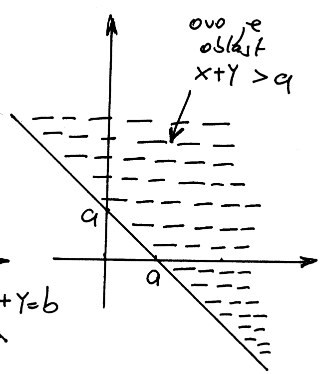
Izračunati integral po oblasti S $\iint_S \frac{1}{xy}\,dx\,dy$ gdje je S oblast ograničena pravama $x+y=a$, $x+y=b$, $y=\alpha x$, $y=\beta x$ gdje su $0<a<b$ i $0<\alpha<\beta$.

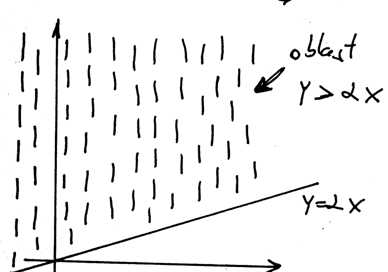
Rj.



Na klasičan način ovaj zadatak nije lagano uraditi. Integral ćemo izračunati uvođenjem smjene. Primjetimo







Iz ovoga možemo primjetiti da je S oblast gdje je $x+y$ između a i b a $\frac{y}{x}$ između α i β.

$$\iint_S \frac{1}{xy}\,dx\,dy = \left|\begin{matrix} u=x+y & a\le u\le b \\ v=\frac{y}{x} & \alpha\le v\le \beta \end{matrix} \qquad J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} \qquad \begin{matrix} y=vx \\ u=x+vx \\ u=x(1+v) \\ x=\frac{u}{1+v} \quad y=\frac{uv}{1+v}\end{matrix}\right.$$

$$\frac{\partial x}{\partial u}=\frac{1}{1+v} \qquad \frac{\partial x}{\partial v}=u\cdot(-1)(1+v)^{-2}=\frac{-u}{(1+v)^2} \qquad dx\,dy=|J|\,du\,dv$$

$$\frac{\partial y}{\partial u}=\frac{v}{1+v} \qquad \frac{\partial y}{\partial v}=\frac{u(1+v)-uv\cdot 1}{(1+v)^2}=\frac{u}{(1+v)^2} \qquad J=\frac{u}{(1+v)^3}+\frac{uv}{(1+v)^3}=\frac{u}{(1+v)^2}\ \Bigg| =$$

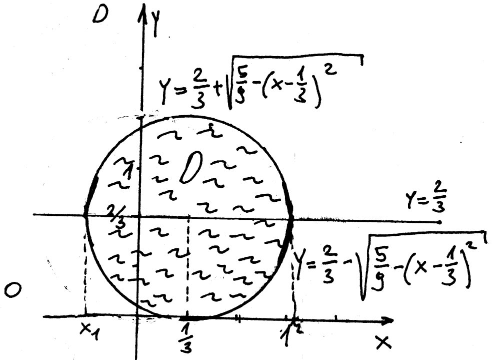
$$=\int_a^b\left[\int_\alpha^\beta \frac{1}{\frac{u}{1+v}\cdot\frac{uv}{1+v}}\cdot\left|\frac{u}{(1+v)^2}\right|dv\right]du=\int_a^b\left[\int_\alpha^\beta \frac{1}{u}\cdot\frac{1}{v}\,dv\right]du=\int_a^b\frac{1}{u}\ln v\Big|_\alpha^\beta=\ln\frac{\beta}{\alpha}\cdot\ln u\Big|_a^b=\ln\frac{\beta}{\alpha}\ln\frac{b}{a}$$

Izračunati dvostruki integral $I=\iint_D (x^2+y^2)\,dx\,dy$ gdje je $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le \frac{2}{3}(x+2y)\}$.

Rj. Odredimo šta je oblast D.

$$x^2+y^2\le\frac{2}{3}(x+2y)$$
$$x^2+y^2\le\frac{2}{3}x+\frac{4}{3}y$$
$$x^2-\frac{2}{3}x+y^2-\frac{4}{3}y\le 0$$
$$x^2-2\cdot x\cdot\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+y^2-2\cdot y\cdot\frac{2}{3}+\frac{4}{9}-\frac{4}{9}\le 0$$
$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2\le\frac{5}{9}$$

D predstavlja unutrašnjost kruga s centrom u tački $\left(\frac{1}{3},\frac{2}{3}\right)$ poluprečnika $r=\frac{\sqrt{5}}{3}\approx 0{,}74$



$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2=\frac{5}{9}$$
$$y=\frac{2}{3}$$
$$\left(x-\frac{1}{3}\right)^2=\frac{5}{9} \qquad x-\frac{1}{3}=\pm\frac{\sqrt{5}}{3} \qquad x_{1,2}=\frac{1\pm\sqrt{5}}{3}$$
$$\left(y-\frac{2}{3}\right)^2=\frac{5}{9}-\left(x-\frac{1}{3}\right)^2 \qquad y=\frac{2}{3}\pm\sqrt{\frac{5}{9}-\left(x-\frac{1}{3}\right)^2}$$

I način: klasičan način

Nađimo presječnu tačku kruga i prave $y=\frac{2}{3}$

$$I=\iint_D (x^2+y^2)\,dx\,dy=\int_{\frac{1-\sqrt{5}}{3}}^{\frac{1+\sqrt{5}}{3}}\left[\int_{\frac{2}{3}-\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}^{\frac{2}{3}+\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}(x^2+y^2)\,dy\right]dx=\ldots$$

NA KLASIČAN NAČIN OVO JE TEŠKO URADITI

II način: Uvedimo neku smjenu promjenjivih. Kako je dat krug uvedimo polarne koordinate.

$$dx\,dy=|\underbrace{J}_{\text{Jakobijan}}|\,dr\,d\varphi$$

$$\begin{matrix} x=a+r\cos\varphi \\ y=b+r\sin\varphi\end{matrix} \quad \text{tj.} \quad \begin{matrix} x=\frac{1}{3}+r\cos\varphi \\ y=\frac{2}{3}+r\sin\varphi\end{matrix} \qquad \begin{matrix} 0\le\varphi\le 2\pi \\ 0\le r\le\frac{\sqrt{5}}{3}\end{matrix}$$

ove vrijednosti čitamo sa slike

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$$dx\,dy=r\,dr\,d\varphi.$$

$$x^2+y^2=\left(\frac{1}{3}+r\cos\varphi\right)^2+\left(\frac{2}{3}+r\sin\varphi\right)^2=\underline{\underline{\frac{1}{9}}}+\frac{2}{3}r\cos\varphi+\underline{r^2\cos^2\varphi}+\underline{\underline{\frac{4}{9}}}+\frac{4}{3}r\sin\varphi+\underline{r^2\sin^2\varphi}=$$
$$=\frac{5}{9}+r^2+\frac{2}{3}r\cos\varphi+\frac{4}{3}r\sin\varphi$$

$$I=\iint_D (x^2+y^2)\,dx\,dy=\iint_{D'}\left(\frac{5}{9}+r^2+\frac{2}{3}r(\cos\varphi+2\sin\varphi)\right)r\,dr\,d\varphi=\iint_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi+$$
$$+\frac{2}{3}\iint_{D'}r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi, \qquad \iint_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{\sqrt{5}}{3}}\left(\frac{5}{9}r+r^3\right)dr\right]d\varphi=2\pi\cdot\left(\frac{5}{9}\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{5}}{3}}+\right.$$

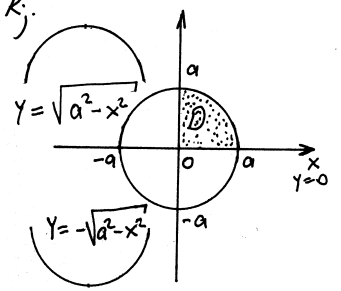
$$+\frac{1}{4}r^4\Big|_0^{\frac{\sqrt{5}}{3}}\Big) = 2\pi\left(\frac{5}{9\cdot 2}\cdot\frac{5}{9}+\frac{1}{4}\cdot\frac{5\cdot 5}{9\cdot 9}\right) = \pi\left(\frac{5^2}{9^2}+\frac{1}{2}\cdot\frac{5^2}{9^2}\right) = \frac{3}{2}\,\frac{5^2}{9^2}\pi = \frac{25}{54}\pi$$

$$\iint_{D'} r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi = \int_0^{\sqrt{5}/3} r^2\left[\int_0^{2\pi}(\cos\varphi+2\sin\varphi)\,d\varphi\right] = \frac{r^3}{3}\Big|_0^{\frac{\sqrt{5}}{3}}\left(\sin\varphi\Big|_0^{2\pi}-2\cos\varphi\Big|_0^{2\pi}\right)=0$$

Prema tome $\iint_D (x^2+y^2)\,dx\,dy = \frac{25}{54}\pi$.

Izračunati $I=\iint_D\sqrt{x^2+y^2}\,dx\,dy$ gdje je D četvrtina kruga $x^2+y^2\le a^2$.

Rj.



$$\iint_D\sqrt{x^2+y^2}\,dx\,dy = \int_0^a\left(\int_0^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\,dy\right)dx =$$

$$= \left|\begin{array}{ll} x=r\cos\varphi & 0\le x\le a \\ y=r\sin\varphi & 0\le y\le\sqrt{a^2-x^2} \\ x^2+y^2=r^2(\cos^2\varphi+\sin^2\varphi)=r^2 & \Downarrow \\ & 0\le\varphi\le\frac{\pi}{2} \\ & 0\le r\le a \end{array}\right.$$

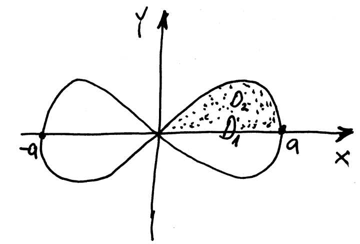
$dx\,dy = |J|\,dr\,d\varphi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix} = \begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix} = r \qquad = \int_0^a\left[\int_0^{\frac{\pi}{2}}\sqrt{r^2}\,|r|\,d\varphi\right]dr$$

$$=\int_0^a\left[\int_0^{\frac{\pi}{2}} r^2\,d\varphi\right]dr = \int_0^a r^2\cdot\varphi\Big|_0^{\frac{\pi}{2}}dr = \int_0^a\frac{\pi}{2}r^2\,dr = \frac{\pi}{2}\cdot\frac{1}{3}r^3\Big|_0^a = \frac{a^3\pi}{6}$$

Izračunati dvostruki integral $\iint_D dx\,dy$, ako je D oblast ograničena lemniskatom $(x^2+y^2)^2=a^2(x^2-y^2)$.

Rj. Lemniskata grafički izgleda ovako.



Pronađimo presječne tačke lemniskate sa x-osom: $y=0$

$x^4=a^2x^2 \Rightarrow x^4-a^2x^2=0$

$x^2(x^2-a^2)=0$

$x_1=0,\ x_2=a,\ x_3=-a$

Primjetimo da se površina oblasti D računa po formuli $P=\iint_D dx\,dy$. Naša oblast D je simetrična u odnosu na y-osu pa je $\iint_D dx\,dy = 2\iint_{D_1}dx\,dy$.

Oblast D_1 je simetrična u odnosu na x-osu.

$\iint_D dx\,dy = 4\iint_{D_2}dx\,dy$ uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$

$(x^2+y^2)^2=a^2(x^2-y^2)$

$(r^2)^2=a^2(r^2\cos^2\varphi-r^2\sin^2\varphi) \quad /:r^2 \ (r\neq 0)$

$r^2=a^2(\cos^2\varphi-\sin^2\varphi)$

$r^2=a^2\cdot\cos 2\varphi \Rightarrow r=a\sqrt{\cos 2\varphi}$

(primjetimo da za $\varphi>\frac{\pi}{4}$ r nije definisano)

$$D_2:\begin{cases}0<\varphi<\frac{\pi}{4}\\ 0<r<\sqrt{a^2\cos 2\varphi}\end{cases}$$

$$\iint_D dx\,dy = 4\iint_{D_2} r\,dr\,d\varphi = 4\int_0^{\frac{\pi}{2}}d\varphi\int_0^{a\sqrt{\cos 2\varphi}} r\,dr = 4\int_0^{\frac{\pi}{4}}\frac{1}{2}r^2\Big|_0^{a\sqrt{\cos 2\varphi}}d\varphi = \frac{4}{2}\int_0^{\frac{\pi}{4}}a^2\cos 2\varphi\,d\varphi =$$

$$=2a^2\cdot\frac{1}{2}\sin 2\varphi\Big|_0^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2}-0\right)=a^2 \quad \text{traženo rješenje}$$

$\#$ Oblast D preslikati pomoću preslikavanja f: $(x, y) \to (u\ v)$.

108. D je oblast ograničena parabolama $y^2 = px$, $y^2 = qx$, $x^2 = ay$, $x^2 = by$, $0 < p < q$, $0 < a < b$, a preslikavanje f je dato jednakostima $y^2 = ux$, $x^2 = vy$.

Rješenja:

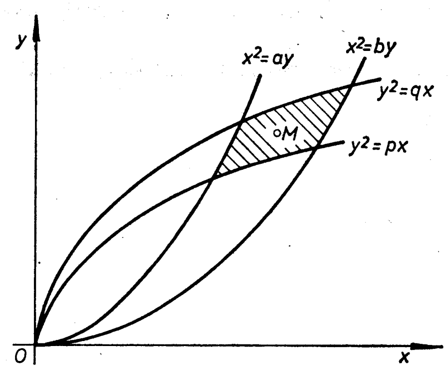
108. Odredićemo Jakobijan preslikavanja. Kako je $u = \dfrac{y^2}{x}$, $v = \dfrac{x^2}{y}$, to je:

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -\dfrac{y^2}{x^2} & \dfrac{2y}{x} \\ \dfrac{2x}{y} & -\dfrac{x^2}{y^2} \end{vmatrix} = 1 - \frac{4xy}{xy} = 1 - 4 = -3.$$

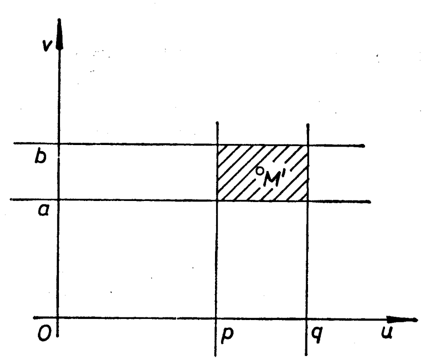
Dakle,

$$J = \frac{D(x, y)}{D(u, v)} = -\frac{1}{3} \neq 0,$$

pa je preslikavanje obostrano jednoznačno. Slike datih parabola su prave $u = p$, $u = q$, $v = a$, $v = b$, a oblast D (sl. 24) se preslikava na pravougaonik D' (sl. 25). Tačka $M \in D$ preslikava se na $M' \in D'$.



Sl. 24



Sl. 25

$\#$ Oblast D preslikati pomoću preslikavanja f: $(x, y) \to (u\ v)$.

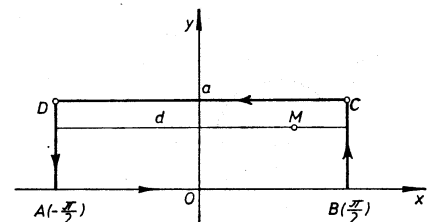
109. $D = \left\{(x, y) : -\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2},\ 0 \leqslant y \leqslant a\right\}$, a preslikavanje f je dato sa $u = \sin x \operatorname{ch} y$, $v = \cos x \operatorname{sh} y$.

Rješenja:

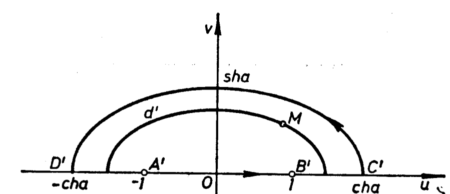
109. Preslikavanje je obostrano jednoznačno na $D \setminus \{A, B\}$, jer je

$$\frac{D(u, v)}{D(x, y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos x \operatorname{ch} y & \sin x \operatorname{sh} y \\ -\sin x \operatorname{sh} y & \cos x \operatorname{ch} y \end{vmatrix} = \cos^2 x \operatorname{ch}^2 y + \sin^2 x \operatorname{sh}^2 y \neq 0$$

za $(x, y) \in D \setminus \{A, B\}$. Dio $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$, $y = 0$ preslikava se na dio $-1 \leqslant u \leqslant 1$, $v = 0$ prave $v = 0$ (sl. 26 i 27). Duž BC ima jednačinu: $x = \dfrac{\pi}{2}$, $0 \leqslant y \leqslant a$, pa je njena slika skup tačaka (u, v) za koje je $u = \operatorname{ch} y$, $v = 0$.



Sl. 26



Sl. 27

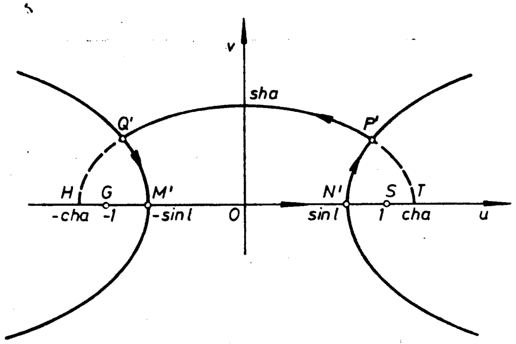
Duž DC ima jednačinu $y = a$, $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$, pa se preslikava na skup tačaka (u, v) za koje je $u = \sin x \operatorname{ch} a$, $v = \cos x \operatorname{sh} a$, $v \geqslant 0$, tj. na gornju polovinu elipse

$$\frac{u^2}{\operatorname{ch}^2 a} + \frac{v^2}{\operatorname{sh}^2 a} = 1.$$

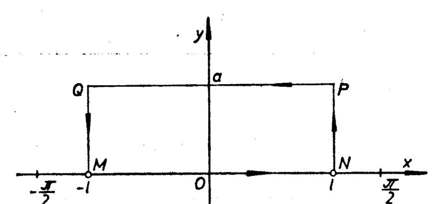
Duž DA preslikava se na duž $D'A'$ (sl. 26 i 27).

Da se unutrašnja tačka M oblasti D preslikava u unutrašnju tačku oblasti gornje poluelipse, može se zaključiti na sljedeći način. Kroz tačku M uočimo duž d paralelnu sa duži AB; njena slika će biti gornji luk elipse čije su poluose manje od $\operatorname{ch} a$ i $\operatorname{sh} a$, pa kako $M \in d \Rightarrow M' \in d'$, to slijedi zaključak.

Primjedba. Neka student sam nađe sliku pravougaonika $D = \{(x, y): -l \leqslant x \leqslant l,\ 0 \leqslant y \leqslant a,\ 0 \leqslant l \leqslant \frac{\pi}{2}$ (sl. 28 a i b). (Prava $x = l$ se preslikava na skup tačaka (u, v) za koje je $u = \sin l \operatorname{ch} y$, $v = \cos l \operatorname{sh} y$, tj. na skup tačaka (u, v) hiperbole $\frac{u^2}{\sin^2 l} - \frac{v^2}{\cos^2 l} = \operatorname{ch}^2 y - \operatorname{sh}^2 y = 1$.)



Sl. 28 a



Sl. 28 b

Kada $l \to \frac{\pi}{2}$, onda figura $M'N'P'Q'$ (sl. 28 a) postaje gornja poluelipsa, tj. $N'P'$ (luk hiperbole) teži duži ST.

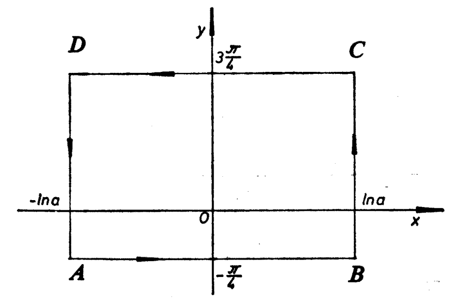
#

Oblast D preslikati pomoću preslikavanja f: $(x, y) \to (u\ v)$.

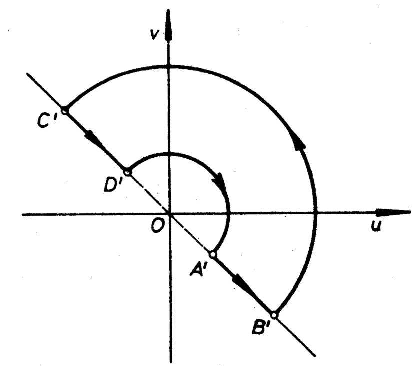
110. $D = \left\{(x, y) : |x| \leqslant \ln a,\ -\frac{1}{4}\pi \leqslant y \leqslant \frac{3}{4}\pi\right\}$, a preslikavanje f je dato sa $u = e^x \cos y$, $v = e^x \sin y$.

Rješenja:

110. Odredimo sliku konture oblasti $\mathscr{D}$ (sl. 29 a).



Sl. 29 a



Sl. 29 b

Duž AB ima jednačinu $y = -\frac{\pi}{4}$, $-\ln a \leqslant x \leqslant \ln a$, pa će biti (sl. 29 b).

$$A'B' = \left\{(u, v) : u = \frac{e^x}{\sqrt{2}},\ v = -\frac{e^x}{\sqrt{2}},\ -\ln a \leqslant x \leqslant \ln a\right\},$$

dakle, $A'B'$ je dio prave $v = -u$, pri čemu je $v < 0$, i to $-\frac{a}{\sqrt{2}} \leqslant v \leqslant -\frac{a^{-1}}{\sqrt{2}}$.

Na isti način zaključujemo da duž CD ima sliku $C'D'$, duž na pravoj $v = -u$,

$$\frac{a^{-1}}{\sqrt{2}} \leqslant v \leqslant \frac{a}{\sqrt{2}}.$$

Duž BC ima jednačinu $x = \ln a$, $-\frac{\pi}{4} \leqslant y \leqslant \frac{3\pi}{4}$, pa će njena slika biti skup tačaka $\{(u, v) : u = a \cdot \cos y,\ v = a \sin y\}$. Dakle, to je dio kružnice poluprečnika a.

Na isti način se zaključuje da duž DA ima kao sliku dio kružnice poluprečnika a^{-1}.

Preslikavanje je obostrano jednoznačno jer je

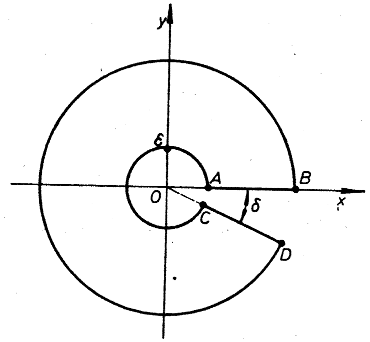
$$\frac{D(u, v)}{D(x, y)} = e^x > 0, \quad \text{za svako } x.$$

\# Oblast D preslikati pomoću preslikavanja $f\colon (x, y) \to (u\ v)$.

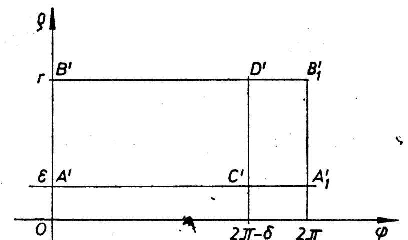
111. $D=\{(x, y): x^2+y^2 \leqslant r^2\}$, a preslikavanje f je dato sa $x=\rho\cos\varphi$, $y=\rho\sin\varphi$.

Rješenja:

111. Kako je $\dfrac{D(x, y)}{D(\rho, \varphi)}=\rho$, a u tački $(0, 0)\in D$ je $\rho=0$, to ćemo najprije naći sliku oblasti $G\subset D$ koja je određena dijelovima kružnica poluprečnika r i ε, dužima AB i CD, pri čemu duž AB leži na x-osi, a duž CD na polupravoj čija je početna tačka $O(0, 0)$ i koja gradi ugao $2\pi-\delta$ (odnosno δ) sa polupravom OB (sl. 30a). Oblast G se preslikava na pravougaonik $A'B'D'C'$ (sl. 31a).



Sl. 30a

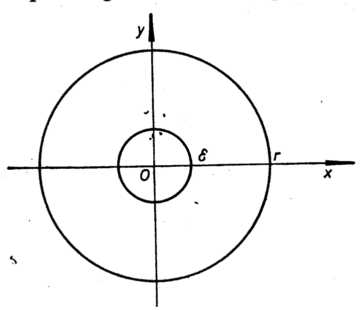


Sl. 31a

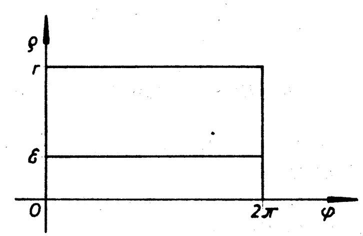
Ako pustimo da $\delta\to 0$, onda tačka $C\to A$, $D\to B$, i $D'\to B_1'$, $C'\to A_1'$.

Dakle, duži AB u ovom preslikavanju odgovaraju i duž $A'B'$ i duž $A_1'B_1'$.

Kružni prsten određen kružnicama poluprečnika r i ε preslikava se na pravougaonik određen pravama $\rho=\varepsilon$, $\rho=r$, $\varphi=0$, $\varphi=2\pi$ (sl. 30b, 31b).



Sl. 30b



Sl. 31b

Ako sada pustimo da $\varepsilon\to 0$, onda slika ε kružnice (duž) teži duži $[0,2\pi]$ na pravoj $\rho=0$ u sistemu $O\rho\varphi$. To znači da u ovom preslikavanju tački $(0, 0)$ odgovara duž $[0,2\pi]$. Krug poluprečnika r preslikava se na pravougaonik $\rho=0$, $\rho=r$, $\varphi=0$, $\varphi=2\pi$.

Primjedba. Neka student uoči značenja veličina ρ i φ u koordinatnom sistemu Oxy.

\# Pomoću smjene promjenljivih izračunati integrale:

114. $\iint\limits_D \sqrt{r^2-(x^2+y^2)}\,dxdy$, gdje je D oblast ograničena kružnicom $x^2+y^2-rx=0$.

115. $\iint\limits_D \ln(x^2+y^2)\,dxdy$, gdje je D oblast ograničena kružnicama $x^2+y^2=e^2$ i $x^2+y^2=e^4$.

Rješenja:

114. Uvodeći smjenu promjenljivih $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, podintegralna funkcija postaje $\sqrt{a^2-\rho^2}$, pa kako je $|J|=\rho$, biće

$$I=\iint\limits_{D'} \sqrt{a^2-\rho^2}\ \rho\,d\rho\,d\varphi.$$

Jednačina kružnice u novim koordinatama je:

$$x^2+y^2-rx=\rho^2\cos^2\varphi+\rho^2\sin^2\varphi-r\rho\cos\varphi=0,$$

tj. $\rho=r\cos\varphi$.

Otuda je

$$I=\int\limits_{-\pi/2}^{\pi/2} d\varphi\int\limits_0^{\rho(\varphi)} \sqrt{r^2-\rho^2}.\ \rho\,d\rho=\int\limits_{-\pi/2}^{\pi/2} d\varphi\int\limits_0^{r\cos\varphi} \rho\sqrt{r^2-\rho^2}\,d\rho=$$

$$=-\frac{1}{3}\int\limits_{-\pi/2}^{\pi/2} (r^2-\rho^2)^{3/2}\Bigg|_0^{r\cos\varphi} d\varphi=\frac{r^3}{3}\int\limits_{+\pi/2}^{\pi/2} (1-\sin^3\varphi)\,d\varphi=\frac{r^3\pi}{3}.$$

115. Smjenom $x=\rho\cos\varphi$, $y=\rho\sin\varphi$ dobija se

$$\iint\limits_D \ln(x^2+y^2)\,dxdy=2\iint\limits_{D'} \rho\ln\rho\ d\rho\ d\varphi=2\int\limits_0^{2\pi} d\varphi\int\limits_e^{e^2} \rho\ln\rho\ d\rho=$$

$$=4\pi\int\limits_e^{e^2} \rho\ln\rho\ d\rho=4\pi\left[\frac{1}{2}\rho^2\ln\rho-\frac{1}{4}\rho^2\right]_e^{e^2}=\pi e^2(3e^2-1).$$

(Za izračunavanje integrala $\int\rho\ln\rho\,d\rho$ primijenjena je parcijalna integracija.)

(#) Pomoću smjene promjenljivih izračunati integrale:

116. $I(r)=\iint\limits_D e^{-x^2-y^2}\,dxdy$, gdje je D oblast ograničena kružnicom $x^2+y^2=r^2$. Naći $\lim I(r)$ kad $r\to\infty$.

117. $\displaystyle\iint\limits_D \frac{dxdy}{(x^2+y^2)\left(1+\sqrt[3]{x^2+y^2}\right)}$. $D=\{(x, y): x^2-y^2\leqslant 0,\ 1\leqslant x^2+y^2\leqslant 4\}$.

Rješenja: **116.** $I(r)=\int\limits_0^{2\pi} d\varphi\int\limits_0^r e^{-\rho^2}\rho\, d\rho=(1-e^{-r^2})\pi$,

$\lim\limits_{r\to\infty} I(r)=\pi$. Ovo znači da je

$$\left(\int\limits_{-\infty}^{\infty} e^{-t^2}\,dt\right)^2=\int\limits_{-\infty}^{\infty} e^{-x^2}dx\int\limits_{-\infty}^{\infty} e^{-y^2}dy=\iint\limits_{-\infty\,-\infty}^{\infty\,\infty} e^{-x^2-y^2}\,dxdy=\pi,$$

tj.

$$\int\limits_{-\infty}^{\infty} e^{-t^2}\,dt=\sqrt{\pi}.$$

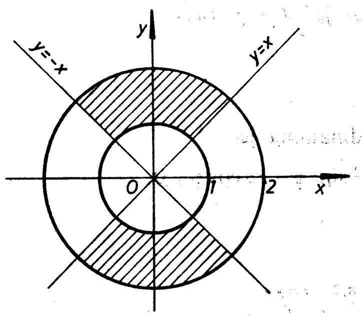
117. Najprije skiciraćemo oblast integracije. Biće:

$\{(x, y): x^2-y^2\leqslant 0\}=\{(x, y): (x-y)(x+y)\leqslant 0\}=\{(x, y): x<y\wedge x>-y$ ili $x>y\wedge x<-y\}$,

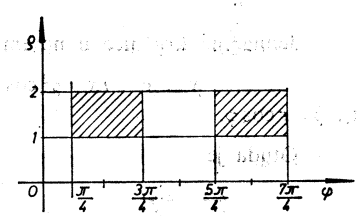
odnosno

$$\{(x, y): x^2-y^2\leqslant 0\}=\{(x, y): x^2\leqslant y^2\}=\{(x, y): |x|\leqslant|y|\}.$$

Oblast integracije D prikazana je na sl. 32a.



Sl. 32a



Sl. 32b

Smjenom $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, data oblast D preslikava se na oblast D' (sl. 32a i b), pa je:

$$I=\iint\limits_{D'}\frac{d\rho\, d\varphi}{\rho\left(1+\sqrt[3]{\rho^2}\right)}=\int\limits_{\pi/4}^{3\pi/4} d\varphi\int\limits_1^2\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}+\int\limits_{5\pi/4}^{7\pi/4} d\varphi\int\limits_1^2\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}=$$

$$=\pi\int\limits_1^2\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}.$$

Smjenom $\sqrt[3]{\rho^2}=t$ dobija se

$$I=\frac{3\pi}{2}\int\limits_1^{\sqrt[3]{4}}\frac{1}{t(t+1)}\,dt=\frac{3\pi}{2}\ln\frac{t}{t+1}\Bigg|_1^{\sqrt[3]{4}}=\frac{\pi}{2}\ln\frac{32}{\left(1+\sqrt[3]{4}\right)^3}.$$

(#) Pomoću smjene promjenljivih izračunati integral:

120. $\iint\limits_D (x+y)^p(x-y)^q\,dxdy$, D je oblast ograničena pravama $x+y=1$, $x-y=1$, $x+y=3$, $x-y=-1$, p realan a q prirodan broj.

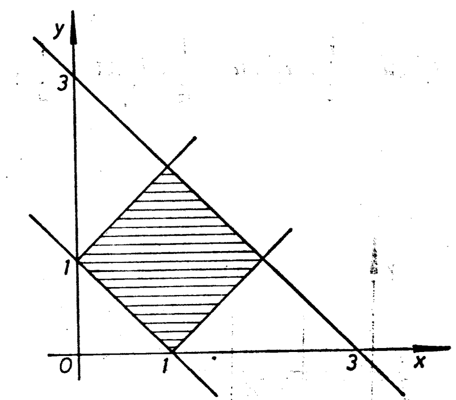
Rješenja:

120. Koristićemo smjenu

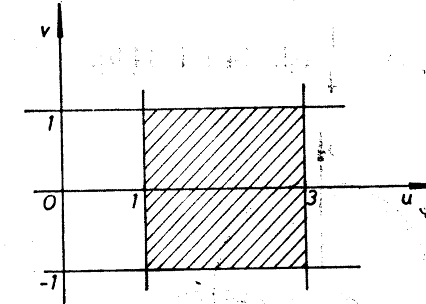
$$x+y=u,\quad x-y=v \Leftrightarrow x=\frac{1}{2}(u+v),\quad y=\frac{1}{2}(u-v).$$

Oblast D (kvadrat na sl. 33a) preslikava se na kvadrat D' (sl. 33b.); preslikavanje je obostrano jednoznačno, jer je

$$J=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\dfrac{1}{2} & \dfrac{1}{2}\\[2mm] \dfrac{1}{2} & -\dfrac{1}{2}\end{vmatrix}=-\frac{1}{2}\ (\neq 0).$$



Sl. 33a



Sl. 33b

Biće: $$I=\iint\limits_{D'} u^p v^q\,|J|\,dudv=\frac{1}{2}\int\limits_1^3 u^p\,du\int\limits_{-1}^1 v^q\,dv=\frac{1}{2}\,\frac{u^{p+1}}{p+1}\Bigg|_1^3\cdot\frac{v^{q+1}}{q+1}\Bigg|_{-1}^1=$$

$$=\frac{1}{2(p+1)(q+1)}\cdot(3^{p+1}-1)\cdot[1-(-1)^{q+1}]\ \text{ za }\ p\neq-1,\quad q\neq-1.$$

Konačno, $I=0$ za $q=2k-1$, $\pm k=1, 2, \ldots$; $I=\dfrac{3^{p+1}-1}{(p+1)(q+1)}$ za $q=2k$, $\pm k=0, 1, 2, \ldots$

Neka student samostalno riješi slučaj $p=-1\vee q=-1$.

Pomoću smjene promjenljivih izračunati integral:

122. $\iint\limits_D (x^2+y^2)^{-2}\,dxdy$, gdje je D oblast ograničena kružnicama $l_1: x^2+y^2-2x=0$, $l_2: x^2+y^2-4x=0$; $l_3: x^2+y^2-2y=0$, $l_4: x^2+y^2-4y=0$.

Rješenja:

122. Napisaćemo jednačine kružnica u obliku

$$l_1: 1-2\frac{x}{x^2+y^2}=0,\quad l_2: 1-4\frac{x}{x^2+y^2}=0,$$

$$l_3: 1-2\frac{y}{x^2+y^2}=0,\quad l_4: 1-4\frac{y}{x^2+y^2}=0$$

i koristićemo smjenu

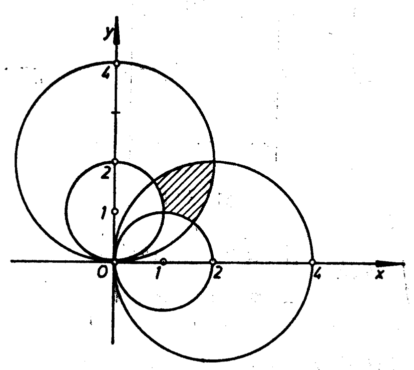
$$\frac{x}{x^2+y^2}=u,\ \frac{y}{x^2+y^2}=v \Leftrightarrow \frac{u}{u^2+v^2}=x,\ \frac{v}{u^2+v^2}=y.$$

Pri tome je

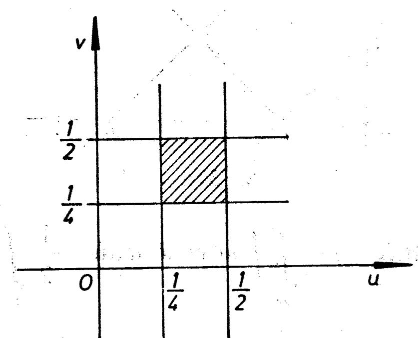
$$\frac{D(x,y)}{D(u,v)}=-\frac{1}{(u^2+v^2)^2},\quad u^2+v^2=\frac{1}{x^2+y^2}.$$

Sada je $\qquad I=\iint\limits_{D'} dudv,$

pri čemu je oblast D' ograničena pravama $l_1': u=\frac{1}{2}$, $l_2': u=\frac{1}{4}$, $l_3': v=\frac{1}{2}$, $l_4': v=\frac{1}{4}$ (sl. 34 a i 34 b).



Sl. 34 a



Sl. 34 b

Biće

$$I=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}.$$

Izračunati dvostruki integral: $I=\iint\limits_D (x+y)\,dxdy$, gdje je

$D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le x+y\}$.

Rj.

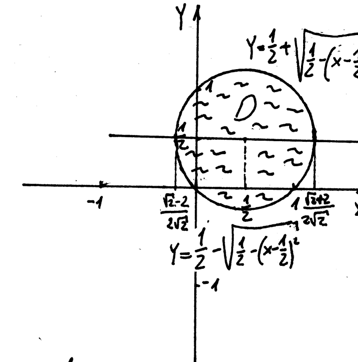
$$x^2+y^2\le x+y$$

$$x^2-x+y^2-y\le 0$$

$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}\le 0$$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{2}$$

Unutrašnjost kruga sa centrom u tački $S\left(\frac{1}{2},\frac{1}{2}\right)$ poluprečnika $r=\frac{1}{\sqrt{2}}\approx 0{,}7$.



Nađimo presječne tačke kruga sa pravom $y=\frac{1}{2}$

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2},\quad y=\frac{1}{2}$$

$$\left(x-\frac{1}{2}\right)^2=\frac{1}{2}\qquad x_1=\frac{1}{2}+\frac{1}{\sqrt{2}}=\frac{2+\sqrt{2}}{2\sqrt{2}}$$

$$x-\frac{1}{2}=\pm\frac{1}{\sqrt{2}}\qquad x_2=\frac{1}{2}-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-2}{2\sqrt{2}}$$

I način:

$$I=\iint\limits_D (x+y)\,dxdy=\int\limits_{\frac{\sqrt{2}-2}{2\sqrt{2}}}^{\frac{\sqrt{2}+2}{2\sqrt{2}}}\left[\int\limits_{\frac{1}{2}-\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}}^{\frac{1}{2}+\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}}(x+y)\,dy\right]dx=\dots$$

$$\left(y-\frac{1}{2}\right)^2=\frac{1}{2}-\left(x-\frac{1}{2}\right)^2$$

$$y-\frac{1}{2}=\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}$$

$$y=\frac{1}{2}\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}$$

KOMPLIKOVANO

II način: Uvedimo neku smjenu promjenljivih.
Kako je u pitanju krug, uvedimo polarne koordinate.

$$x=a+r\cos\varphi \qquad \text{tj.}\qquad x=\frac{1}{2}+r\cos\varphi \qquad 0\le\varphi\le 2\pi$$

$$y=b+r\sin\varphi \qquad\qquad y=\frac{1}{2}+r\sin\varphi \qquad 0\le r\le\frac{\sqrt{2}}{2}$$

ove vrijednosti čitamo sa slike

Jakobijan

$$dxdy=|J|\,drd\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r.$$

$$dxdy=r\,drd\varphi$$

$$I=\iint\limits_D (x+y)\,dxdy=\iint\limits_{D'}\left(\frac{1}{2}+r\cos\varphi+\frac{1}{2}+r\sin\varphi\right)r\,drd\varphi=\iint\limits_{D'}\left(r+r^2(\cos\varphi+\sin\varphi)\right)drd\varphi$$

$$=\iint\limits_{D'} r\,drd\varphi+\iint\limits_{D'} r^2(\cos\varphi+\sin\varphi)\,drd\varphi,\quad \iint\limits_{D'} r\,drd\varphi=\int\limits_0^{2\pi}\left[\int\limits_0^{\frac{\sqrt{2}}{2}} r\,dr\right]d\varphi=2\pi\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{2}}{2}}=\frac{\pi}{2}$$

$$\iint_{0} r^2(\cos\varphi+\sin\varphi)\,dr\,d\varphi=\int_0^{2/\sqrt{2}} r^2\left[\int_0^{2\pi}(\cos\varphi+\sin\varphi)\,d\varphi\right]dr=\frac{1}{3}r^3\Big|_0^{\frac{2}{\sqrt{2}}}\cdot\left(\sin\varphi\Big|_0^{2\pi}-\cos\varphi\Big|_0^{2\pi}\right)$$

$$=\frac{1}{3}\cdot\frac{8}{2\sqrt{2}}\left(0-(1-1)\right)=0$$

Prema tome:

$$\iint_{0}(x+y)\,dx\,dy=\frac{\pi}{2}$$

Zadaci za vježbu

U zadacima 3525 — 3531 transformisati dvojni integral $\iint_D f(x,\ y)\,dx\,dy$ na polarne koordinate ρ i φ ($x=\rho\cos\varphi$, $y=\rho\sin\varphi$), a zatim ga svesti na dvostvruki (sa određenim posebnim granicama integracije).

3525. D je krug: 1) $x^2+y^2<R^2$; 2) $x^2+y^2<ax$; 3) $x^2+y^2<by$.

3526. D je oblast ograničena kružnim linijama $x^2+y^2=4x$, $x^2+y^2=8x$ i pravama $y=x$ i $y=2x$.

3527. D je oblast koja predstavlja zajednički deo dva kruga $x^2+y^2<ax$ i $x^2+y^2<by$.

3528. D je oblast ograničena pravama $y=x$, $y=0$ i $x=1$.

3529. D je odsečak koji prava $x+y=2$ odseca od kruga $x^2+y^2=4$.

3530. D je oblast ograničena desnom petljom lemniskate $(x^2+y^2)^2=a^2(x^2-y^2)$.

3531. D je oblast određena nejednakostima $x>0$, $y\geqslant 0$, $(x^2+y^2)^3<4a^2x^2y^2$.

U zadacima 3532 — 3535 date dvostruke integrale transformisati na polarne koordinate.

3532. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}} f(x,\ y)\,dy$. **3533.** $\int_{\frac{R}{2}}^{2R} dy\int_0^{\sqrt{2Ry-y^2}} f(x,\ y)\,dx$.

3534. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}} f(x^2+y^2)\,dy$.

3535. $\int_0^{\frac{R}{\sqrt{1+R^2}}} dx\int_0^{Rx} f\left(\frac{y}{x}\right)dy+\int_{\frac{R}{\sqrt{1+R^2}}}^{R} dx\int_0^{\sqrt{R^2-x^2}} f\left(\frac{y}{x}\right)dy$.

U zadacima 3536 — 3540 izračunati date dvojne integrale prelazeći na polarne koordinate.

3536. $\iint_D \ln(1+x^2+y^2)\,dx\,dy$, oblast D je četvrtina kruga $x^2+y^2<R^2$ koja leži u prvom kvadrantu.

3537. $\iint_D \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, oblast D je određena nejednakostima $x^2+y^2<1$, $x>0$, $y>0$.

3538. $\iint_D (h-2x-3y)\,dx\,dy$, D je krug $x^2+y^2<R^2$.

3539. $\iint_D \sqrt{R^2-x^2\quad y^2}\,dx\,dy$, D je krug $x^2+y^2<Rx$.

3540. $\iint_D \operatorname{arctg}\frac{y}{x}\,dx\,dy$, D je deo prstena $x^2+y^2>1$, $x^2+y^2<9$, $y\geqslant\frac{x}{\sqrt{3}}$, $y<x\sqrt{3}$.

Rješenja

3525. 1) $\int_0^{2\pi} d\varphi\int_0^{0} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$;

2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{a\cos\varphi} f(\rho\cos\varphi,\ \rho\sin\rho)\,\rho\,d\rho$;

3) $\int_0^{\pi} d\varphi\int_0^{b\sin\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3526. $\int_{\frac{\pi}{4}}^{\operatorname{arctg}2} d\varphi\int_{4\cos\varphi}^{8\cos\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3527. $\int_0^{\operatorname{arctg}\frac{a}{b}} d\varphi\int_0^{b\sin\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho+$

$\int_{\operatorname{arctg}\frac{a}{b}}^{\frac{\pi}{2}} d\varphi\int_0^{a\cos\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3528. $\int_0^{\frac{\pi}{4}} d\varphi\int_0^{\sec\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3529. $\int_0^{\frac{\pi}{2}} d\varphi\int_{\sqrt{2}\sec\left(\varphi-\frac{\pi}{4}\right)}^{2} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3530. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi\int_0^{a\sqrt{\cos 2\varphi}} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3531. $\int_0^{\frac{\pi}{2}} d\varphi\int_0^{a\sin 2\varphi} f(\rho\cos\varphi),\ \rho\sin\varphi)\,\rho\,d\rho$.

3532. $\int_0^{\frac{\pi}{2}} d\varphi\int_0^{R} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3533. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi\int_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3541. Na osnovu geometrijskih razmatranja pokazati da: ako se dekartove kordinate transformišu shodno obrascima $x=a\rho\cos\varphi$, $y=b\rho\sin\varphi$, u kojima su a i b konstante, onda će elemenat površine biti $d\sigma=ab\,\rho\,d\rho\,d\varphi$.

U zadacima 3542 — 3544 koristeći rezultat prethodnog zadatka i izabravši najpogodnije vrednosti za a i b, transformisati dvojne integrale.

3542. $\iint_D f(x,\ y)\,dx\,dy$. D je oblast ograničena elipsom $\frac{x^2}{4}+\frac{y^2}{9}=1$.

3543. $\iint_D f(x,\ y)\,dx\,dy$. D je oblast ograničena krivom $\left(x^2+\frac{y^2}{3}\right)^2=x^2y$.

3544. $\iint_D f\left(\sqrt{4-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dx\,dy$, D je deo eliptičnog prstena ograničenog elipsama $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ i $\frac{x^2}{4a^2}+\frac{y^2}{4b^2}=1$, koji leži u prvom kvadrantu.

3545. Izračunati integral $\iint_D xy\,dx\,dy$, u kojem je D oblast u prvom kvadrantu, ograničena elipsom $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3546. Izračunati integral $\iint_D \sqrt{xy}\,dx\,dy$, u kojem je D oblast u prvom kvadrantu, ograničena krivom $\left(\frac{x^2}{2}+\frac{y^2}{b}\right)^4=\frac{xy}{\sqrt{6}}$.

Rješenja

3534. $\frac{\pi}{2}\int_0^R f(\rho^2)\,\rho\,\rho d$. **3535.** $\frac{R^2}{2}\int_0^{\operatorname{arctg}R} f(\operatorname{tg}\varphi)\,d\varphi$.

3536. $\frac{\pi}{4}[(1+R^2)\ln(1+R^2)-R^2]$. **3537.** $\frac{\pi(\pi-2)}{8}$. **3538.** $\pi R^2 h$.

3539. $\frac{R^3}{3}\left(\pi-\frac{4}{3}\right)$. **3540.** $\frac{\pi^2}{6}$.

3542. $x=2\rho\cos\varphi$, $y=3\rho\sin\varphi$; $I=6\int_0^{2\pi} d\varphi\int_0^{1} f(2\rho\cos\varphi,\ 3\rho\sin\varphi)\,\rho\,d\rho$.

3543. $x=\rho\cos\varphi$, $y=\sqrt{3}\,\rho\sin\varphi$;

$$I=\sqrt{3}\int_0^{\pi} d\varphi\int_0^{\sqrt{3}\cos^2\varphi\sin\varphi} f(\rho\cos\varphi,\ \sqrt{3}\,\rho\sin\varphi)\,\rho\,d\rho.$$

3544. $x=a\rho\cos\varphi$, $y=b\rho\sin\varphi$; $I=ab\int_0^{\frac{\pi}{2}} d\varphi\int_1^{2} f(\sqrt{4-\rho^2})\,\rho\,d\rho$.

3545. $\frac{a^2b^2}{8}$. **3546.** $\frac{1}{\frac{4}{\sqrt{6}}}$.